UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2007 1.30 to 4.30

PAPER 62

COSMOLOGY

*Attempt **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a flat FRW universe containing only matter and a cosmological constant.

(a) Show that Friedmann's equation may be written

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_M}{a^3} + (1-\Omega_M)\right), \qquad (*)$$

where $a(t)$ is the scale factor normalised to unity today. Explain the physical significance of the parameters H_0 and Ω_M.

(b) By changing variables to $x = a^{\frac{3}{2}}$, integrate $(*)$ to obtain

$$a(t) = \left(\frac{\Omega_M}{1-\Omega_M}\right)^{\frac{1}{3}}\left(\sinh\left(\tfrac{3}{2}H_0\sqrt{1-\Omega_M}\,t\right)\right)^{\frac{2}{3}}.$$

Describe the dependence of the age of the universe on Ω_M, (i) as Ω_M tends to zero and (ii) as Ω_M tends to unity.

(c) Show that the furthest comoving distance an observer will *ever* be able to see in such a universe, if they can see all the way back to the initial singularity, is given by:

$$d_c = H_0^{-1}\int_0^\infty \frac{dy}{\sqrt{(1-\Omega_M)+\Omega_M y^3}}.$$

[*Hint: change variables from t to a and then to $y = 1/a$.*]

2 The number density n_A of particle species A in thermal equilibrium at a temperature $T \ll M_A$ is given by

$$n_A = g_A \left(\frac{M_A T}{2\pi} \right)^{\frac{3}{2}} e^{(\mu_A - M_A)/T} .$$

(a) Explain the meaning of the constants g_A and μ_A. What determines μ_A?

(b) Consider protons, electrons and hydrogen atoms around the time of "recombination", $T \approx 3000°\text{K} \approx 0.3$ eV. Show that

$$\frac{n_H}{n_p n_e} \approx \left(\frac{2\pi}{M_e T} \right)^{\frac{3}{2}} e^{(B/T)} ,$$

where $B = M_P + M_e - M_H \approx 13.6$ eV.

(c) Using charge neutrality, $n_p = n_e$, and the fact that the baryon asymmetry $\eta \approx (n_H + n_P)/n_\gamma$ is very small, explain why the temperature at which the protons and electrons combine into neutral hydrogen is approximately given by

$$T \approx \frac{B}{\ln \left(\eta^{-1} (M_e/T)^{\frac{3}{2}} \right)} \approx \frac{B}{\ln \left(\eta^{-1} (M_e/B)^{\frac{3}{2}} \right)} \ll B, \qquad (*)$$

where factors of order unity are ignored inside the logarithm. Using $M_e = 0.5$ MeV and $\eta = 10^{-10}$, check that (∗) gives a reasonable estimate of the temperature of recombination.

3 Consider a simple inflationary model consisting of a scalar field with potential $V(\phi)$ in a flat FRW universe. The fluctuations of the scalar field, $\delta\phi$, may be treated in the first approximation as a massless field in a background flat FRW spacetime with Hubble parameter $H^2 = 8\pi G V(\phi)/3 \approx \text{const}$.

(a) Show that in this approximation the background metric may be written in conformal time as $a^2(\tau)(-d\tau^2 + d\mathbf{x}^2)$ with $a(\tau) = -1/(H\tau)$, $-\infty < \tau < 0$.

(b) The equation of motion for a massless field in a curved spacetime is

$$\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu\right) \delta\phi = 0. \qquad (*)$$

Show that in given background, if we set $\delta\phi(\tau, \mathbf{x}) = \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, this becomes

$$\delta\phi''_{\mathbf{k}} - \frac{2}{\tau}\delta\phi'_{\mathbf{k}} = -k^2 \delta\phi_{\mathbf{k}}. \qquad (**)$$

(c) Show that $(**)$ is solved by

$$\delta\phi^{(+)}_{\mathbf{k}}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k\mathcal{V}}} H \left(-\tau + \frac{i}{k}\right),$$

where $\mathcal{V}$ is a large comoving volume, so that $\sum_{\mathbf{k}}$ may be replaced by $\mathcal{V} \int d^3\mathbf{k}/(2\pi)^3$. Describe the physical significance of this solution.

(d) The quantum field may be expressed as

$$\hat{\delta\phi}(\tau, \mathbf{x}) = \sum_{\mathbf{k}} \left(a_{\mathbf{k}} \delta\phi^{(+)}_{\mathbf{k}}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + h.c.\right)$$

where $h.c.$ denotes hermitian conjugate. Using the commutation relations $[a_{\mathbf{k}}, {a_{\mathbf{k}'}}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}$, and $a_{\mathbf{k}}|0, \text{in}\rangle = 0$, show that the variance of the quantum field in the incoming Minkowski vacuum $|0, \text{in}\rangle$ is formally given by

$$\langle 0, \text{in}|\hat{\delta\phi}^2(t, \mathbf{x})|0, \text{in}\rangle = \sum_{\mathbf{k}} \frac{1}{2k\mathcal{V}} \left(\frac{1}{a^2} + \frac{H^2}{k^2}\right) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} \left(\frac{1}{a^2} + \frac{H^2}{k^2}\right).$$

Interpret each term and hence explain why, as $\tau \to 0$ and a becomes very large, the scalar field acquires a scale-invariant pattern of frozen-in fluctuations.

4 Consider an inflationary model in which a scalar field ϕ rolls slowly down a potential $V(\phi)$, acquiring fluctuations $\delta\phi \approx H/(2\pi)$ on each comoving wavenumber k as the corresponding physical wavelength $\lambda = a(2\pi/k)$ exits the Hubble radius H^{-1} during inflation.

(a) Briefly explain how the fluctuations generate a space-dependent time delay $\delta t = \delta\phi/\dot{\phi}$, and how this eventually leads to a growing mode density perturbation of magnitude $\delta\rho/\rho \approx H\delta t$. From the spectrum of frozen-in scalar field fluctuations, $\langle\delta\phi^2\rangle \approx (H/2\pi)^2 \int(dk/k)$, explain why the final spectrum of growing-mode density perturbations

$$\langle(\delta\rho/\rho)^2\rangle \approx \int \frac{dk}{k}\left(\frac{H^2}{2\pi\dot{\phi}}\right)^2, \qquad (*)$$

where the integrand is to be evaluated as each scale k leaves the Hubble radius.

(b) As a given wavenumber exits the Hubble radius, we have $k \approx (2\pi H)a$, and we can treat H as slowly-varying. Using this relation to relate k to a, and the slow-roll approximation for the scalar field and Friedmann equations to relate a to ϕ, show that

$$k\frac{d}{dk} \approx a\frac{d}{da} \approx \frac{\dot{\phi}}{H}\frac{d}{d\phi} \approx -\frac{V_{,\phi}}{V}\frac{d}{d\phi}.$$

in units where $8\pi G = 1$.

(c) Hence show using $(*)$ that the spectral tilt Δn, *i.e.*, the deviation from scale-invariance, defined by

$$\langle(\delta\rho/\rho)^2\rangle = \mathcal{C}\int \frac{dk}{k}k^{\Delta n},$$

with $\mathcal{C}$ a constant, is given by

$$\Delta n = -6\epsilon + 2\eta,$$

where the slow-roll parameters $\epsilon \equiv \frac{1}{2}(V_{,\phi}/V)^2$ and $\eta \equiv V_{,\phi\phi}/V$, again in units where $8\pi G = 1$.

5 Consider linearised density perturbations of comoving wavenumber k in a flat FRW universe containing only cold dark matter and radiation. In synchronous coordinates which are comoving with the CDM, the linearised Einstein equations yield

$$\delta_C'' + \frac{a'}{a}\delta_C' = 4\pi G a^2(\rho_C \delta_C + 2\rho_R \delta_R), \tag{1}$$

where ρ_C and δ_C are the energy density and fractional energy density perturbation for the CDM, ρ_R and δ_R are the same quantities for the radiation, and primes denote conformal time (τ) derivatives.

(a) In the early universe, the radiation dominates. Show, using the Friedmann equation, that $a(\tau) \propto \tau$ and $8\pi G \rho_R a^2 = 3\tau^{-2}$. The CDM density is negligible compared to that of radiation, but we are still interested in tracking the fractional density perturbation in the CDM. Show that, if we choose adiabatic initial conditions $\delta_R = \frac{4}{3}\delta_C$ on large scales $k\tau \ll 1$, then (1) possess the growing solution

$$\delta_C(\tau, \mathbf{k}) = A_R(\mathbf{k})\tau^2. \tag{2}$$

(b) Consider Fourier modes which are initially in the adiabatic growing solution (2), for $k\tau \ll 1$. When these modes enter the Hubble radius, i.e., when $k\tau$ grows larger than unity, in the radiation epoch, δ_R oscillates and hence averages to zero. Show that in this regime, where δ_R is negligible and $\rho_C \ll \rho_R$, the relevant solution of (1), with initial conditions given by (2), is

$$\delta_C(\tau, \mathbf{k}) \approx A_R(\mathbf{k}) k^{-2} \ln(k\tau). \tag{3}$$

(c) In the late universe, the radiation may be neglected. Show, using the Friedmann equation, that the scale factor $a(\tau) \propto \tau^2$ and $8\pi G \rho_C a^2 = 12\tau^{-2}$. Hence show that (1) possesses the growing solution

$$\delta_C(\tau, \mathbf{k}) \propto \tau^2, \tag{4}$$

on all scales.

END OF PAPER