UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2007 1.30 to 4.30

PAPER 61

GENERAL RELATIVITY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

Information

The signature is $(+ - - -)$*, and the curvature tensor conventions are defined by*

$$R^i{}_{kmn} = \Gamma^i{}_{km,n} - \Gamma^i{}_{kn,m} - \Gamma^i{}_{pm}\Gamma^p{}_{kn} + \Gamma^i{}_{pn}\Gamma^p{}_{km}\,.$$

Units are chosen so that $c = 1$*.*

You may use freely without proof or justification in your answer to any question, the information given at the end of questions 1 and 2 and also that on the two-sided lecture handout attached to this paper.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
Lecture handout.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A covector n_a is normal to the surface $\beta(x) = 0$. Show that

$$n_{[a;b}n_{c]} = 0.$$

Such covectors are said to be *hypersurface-orthogonal.*

Show that a Killing covector k_a satisfies

$$k_{a;bc} = R_{abc}{}^{d}k_d.$$

A spacetime is said to be *static* if it possesses a timelike Killing covector k_a which is hypersurface-orthogonal. By considering $(k_{[a;b}k_{c]})^{;c}$, or otherwise, show that

$$k_d R^d{}_{[a}k_{b]} = 0.$$

Deduce that if a static spacetime is filled by a perfect fluid of energy density ρ and pressure p with $\rho + p > 0$ then the fluid 4-velocity u_a is parallel to k_a.

[*The Ricci identity is* $X_{a;bc} - X_{a;cb} = R_a{}^d{}_{bc}X_d$.]

2 Comment briefly on Einstein's *Strong Equivalence Principle* (SEP) and the "comma goes to semicolon rule", emphasizing the physical consequences.

In electromagnetism in Minkowski spacetime the equations relating the 4-vector potential A^a, the Maxwell field tensor F_{ab} and the 4-current density J^a are

$$F_{ab} = A_{b,a} - A_{a,b}, \qquad F^{ab}{}_{,b} = -\mu_0 J^a,$$

where μ_0 is a constant. Show that

$$A^{a,b}{}_b - A^{b,a}{}_b = \mu_0 J^a, \tag{1}$$

and

$$A^{a,b}{}_b - A^b{}_{,b}{}^a = \mu_0 J^a. \tag{2}$$

Discuss the application of the SEP to this theory. Show that the generalization to a curved vacuum spacetime is unambiguous.

Could a terrestial non-vacuum experiment resolve the ambiguity in a non-vacuum spacetime?

[*In units where* $c = 1$, $G = 7\times10^{-29} cm\ g^{-1}$, *and the radius of the Earth is* $R = 6\times10^8 cm$.]

3 Consider linearized perturbations of Minkowski spacetime with a weak field source $\delta T_{ab}(t, \mathbf{x})$ so that

$$\delta G_{ab} = -8\pi G\, \delta T_{ab}.$$

Using the notation and information from the lecture handout included with this examination paper and considering only tensor perturbations show that

$$\ddot{E}^{\alpha\beta} - \Delta E^{\alpha\beta} = 8\pi G\, \delta\widehat{T}^{\alpha\beta}, \qquad (*)$$

where

$$\delta\widehat{T}^{\alpha\beta} = \delta T^{\alpha\beta} - \tfrac{1}{3}\gamma^{\alpha\beta}\delta T^{\delta}{}_{\delta}.$$

You may assume that the standard *retarded potential* solution of equation (*) is

$$E^{\alpha\beta}(t, \mathbf{x}) = 2G \int \frac{\delta\widehat{T}^{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x',$$

and that far away from a compact source, $|\mathbf{x}'| \ll |\mathbf{x}| = r$,

$$E^{\alpha\beta}(t, \mathbf{x}) = \frac{2G}{r} \int \delta\widehat{T}^{\alpha\beta}(t - r, \mathbf{x}')\, d^3x'.$$

Here all integrals are taken over the source.

Show that

$$E^{\alpha\beta}(t, \mathbf{x}) = \frac{G}{r}\ddot{Q}^{\alpha\beta}(t - r),$$

where

$$Q^{\alpha\beta}(t) = \int \delta T^{00}(t, \mathbf{x}')[x'^{\alpha}x'^{\beta} - \tfrac{1}{3}\gamma^{\alpha\beta}x'^{\delta}x'_{\delta}]\, d^3x'$$

is the *quadrupole moment* of the compact source.

Suppose the source has mass M and spatial extent R, and moves slowly under its own weak gravitational field. Estimate the order of magnitude of $|E^{\alpha\beta}|$ in terms of r, R and $\epsilon = GM/R \ll 1$.

4 Let $d\Sigma^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ denote the line element on the unit 2-sphere, and let

$$\widehat{ds}^2 = dt^2 - dr^2 - r^2\, d\Sigma^2$$

denote the Minkowski spacetime line element in spherical polar coordinates. Introduce retarded coordinates $u = t - r$ and $v = t + r$ and make a conformal transformation to an unphysical spacetime with line element ds^2 given by $ds^2 = 4(1 + u^2)^{-1}(1 + v^2)^{-1}\widehat{ds}^2$. Perform further coordinate changes $p = \tan^{-1} u$, $q = \tan^{-1} v$, followed by $T = q + p$, $R = q - p$ to obtain

$$ds^2 = dT^2 - dR^2 - \sin^2 R\, d\Sigma^2,$$

where the ranges of all of the coordinates t, r, θ, ϕ, u, v, p, q, T and R should be stated explicitly.

Use your results to discuss the asymptotic behaviour of Minkowski spacetime geodesics as seen in the unphysical spacetime. What, if any, horizon structure is there?

END OF PAPER