UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 12 June 2007 1.30 to 3.30

PAPER 60

CONTROL OF QUANTUM SYSTEMS

Attempt ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Generalized Bloch vectors. The generalized Bloch vector (also called coherence vector or Stokes tensor) of a density operator $\hat{\rho}$ for an N-dimensional system with $N > 2$ is defined (analogous to the $N = 2$ case) by $s_k = \mathrm{Tr}[\hat{\rho}\hat{\sigma}_k]$, $k = 1, \ldots, N^2 - 1$, where $\hat{\sigma}_k$ are the generalized Pauli matrices

$$\hat{\sigma}^x_{rs} = (|r\rangle\langle s| + |s\rangle\langle r|)/\sqrt{2},$$

$$\hat{\sigma}^y_{rs} = i(-|r\rangle\langle s| + |s\rangle\langle r|)/\sqrt{2}$$

$$\hat{\sigma}^z_r = \sqrt{\tfrac{1}{r+r^2}}\left(\sum_{k=1}^{r} |k\rangle\langle k| - r|r+1\rangle\langle r+1|\right)$$

for $1 \le r \le N-1$ and $r < s \le N$.

(a) Using $\mathrm{Tr}[\hat{\sigma}_k\hat{\sigma}_\ell] = \delta_{k\ell}$, which you do *not* need to prove, show that the generalized Pauli matrices $\{\hat{\sigma}_k\}_{k=1}^{N^2-1}$ form an orthonormal basis for the $N \times N$ trace-zero Hermitian matrices, and that $\{\hat{\sigma}_k\}_{k=0}^{N^2-1}$ is an orthonormal basis for all $N \times N$ Hermitian matrices. Furthermore, show that the generalized Bloch vectors are elements in $\mathbb{R}^{N^2-1}$.

(b) Show that pure states in $\mathcal{C}^N$ correspond to points on the sphere S^{N^2-2} of radius $\sqrt{1-1/N}$ in this space.

(c) Show that the set of pure states does not cover the sphere S^{N^2-2} for $N > 2$.

(d) Show that non-pure states lie in the interior of the sphere S^{N^2-2} (of radius $\sqrt{1-1/N}$) for $N \ge 2$.

(e) Show that Hamiltonian evolution of a general quantum ensemble corresponds to a rotation of the generalized Bloch vector.

In the presence of dissipation the generalized Bloch vector satisfies a dissipative Bloch equation $\dot{\mathbf{s}}(t) = A\mathbf{s} + \mathbf{c}$, and for $N = 2$ we have explicitly

$$A = \begin{pmatrix} -\Gamma & -\alpha_z & \alpha_y \\ \alpha_z & -\Gamma & -\alpha_x \\ -\alpha_y & \alpha_x & -(\gamma_{12}+\gamma_{21}) \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ \gamma_{21} - \gamma_{12} \end{pmatrix},$$

where α_k for $k = x, y, z$ are control parameters and Γ, γ_{12}, γ_{21} relaxation parameters.

(f) What type of relaxation does each of the relaxation parameters in the dissipative Bloch equation correspond to? What are the steady states of the system in the absence of any control fields, i.e. for $\alpha_x = \alpha_y = \alpha_z = 0$?

2 Controllability.

(a) Explain *briefly* what a bilinear Hamiltonian control system is, and give *concise* general definitions of the control-theoretic notions of *reachable set* and *controllability.*

(b) Define what a Lie algebra is, and give Lie-algebraic criteria for the notions of pure-state, mixed-state and unitary operator controllability for a bilinear Hamiltonian control system.

(c) Show that the dynamical Lie group generated by $i\hat{H}_0$ and $i\hat{H}_1$ for a four-level system governed by the control Hamiltonian $\hat{H}[f(t)] = \hat{H}_0 + f(t)\hat{H}_1$ satisfies a sympletic symmetry with respect to the symmetry operator $\hat{J}$, where $(\alpha, \beta \in \mathbb{R})$

$$\hat{H}_0 = \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}, \; \hat{H}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; \hat{J} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

(d) Show that the (unitarily equivalent) states $\hat{\rho}_0$ and $\hat{\rho}_1$ below are not dynamically equivalent under the action of the Lie group defined in part (c).

$$\hat{\rho}_0 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \hat{\rho}_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad 0 \le a < \tfrac{1}{2}, b = \tfrac{1}{2} - a, b \neq a.$$

(e) Briefly *outline* a proof showing that density matrix controllability implies pure state controllability, and indicate whether the converse is true or not (i) in general and (ii) for the special case of $N = 2$.

3 Geometric control and applications.

(a) If $\hat{A}$ is a bounded operator on a Hilbert space such that $\hat{A}^2 = \hat{I}$, where $\hat{I}$ is the identity matrix, show that $\exp(-i\theta\hat{A}) = \cos(\theta)\hat{I} - i\sin(\theta)\hat{A}$.

(b) Show that a generic single-qubit quantum gate can be expressed as

$$\hat{U} = \exp\left[-\tfrac{i\theta}{2}(n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z)\right],$$

where $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices.

(c) Use the results from part (b) to outline a scheme for implementing single qubit gates for a qubit subject to the control Hamiltonian $H_1[\mathbf{B}(t)] = B_x(t)\hat{\sigma}_x + B_y(t)\hat{\sigma}_y$.

Next consider two qubits, each with a single qubit Hamiltonian of the form $\hat{H}_1$ above, and Ising coupling, i.e., assume the total Hamiltonian is

$$H = B_x^{(1)}(t)\,\hat{\sigma}_x \otimes \hat{I} + B_y^{(1)}(t)\,\hat{\sigma}_y \otimes \hat{I} + B_x^{(2)}(t)\,\hat{I} \otimes \hat{\sigma}_x + B_y^{(2)}(t)\,\hat{I} \otimes \hat{\sigma}_y + H_I$$

and $H_I = J_{12}\,\hat{\sigma}_z \otimes \hat{\sigma}_z$, where the fields $B_x^{(k)}(t)$, $B_y^{(k)}(t)$, $k = 1, 2$ and the inter-qubit coupling J_{12} are control inputs.

(d) Suggest a simple scheme to implement a two-qubit phase gate of the form $C_{\text{phase}} = \text{diag}(-1, 1, 1, -1)$.

(e) Explain how you could implement an arbitary two-qubit gate using only C_{phase} gates and single qubit x, y or z rotations. (Hint: Use the Cartan decomposition.)

(f) If the inter-qubit coupling J_{12} is fixed, i.e., cannot be controlled, what problems will arise? Under what conditions can the coupling be neglected?

4 Control field design. Given a (controllable) system, explain how we can design control fields to realize a particular control objective using open-loop, model-based control design techniques. Briefly outline three design strategies in one or two sentences each, and then explain one technique in detail. You should include relevant equations, algorithms, etc.

For instance, if you choose variational optimal control, you should include the variational functional, the Euler-Lagrange equations, sketch an algorithm to solve the equations, and comment on how the resulting control fields can be implemented using pulse shaping techniques.

5 Stochastic feedback control.

(a) Write down the interaction Hamiltonian for a cavity with a quantized field, and show that the rotating wave approximation (RWA) leads to

$$\hat{H}_I(t) = i\sqrt{\gamma}[\hat{a}(t) \otimes \hat{b}^\dagger(t) - \hat{a}^\dagger(t) \otimes \hat{b}(t)],$$

where $\hat{a}^\dagger$ is the cavity creation operator, $\hat{b}^\dagger$ the creation operator for the external field, and γ is the cavity-field coupling strength.

(b) Let $\hat{b}_{\rm in}(t)$ and $\hat{b}_{\rm out}(t)$ be stochastic input and output operators, respectively, for the cavity, and $\hat{B}_{in}(t) = \int_t^{t+dt} \hat{b}(\tau)\,d\tau$. Show that, assuming Markovian dynamics, the stochastic evolution operator is

$$\hat{U}_I(t+dt,t) = \exp\left[\sqrt{\gamma}[\hat{a}(t) \otimes \hat{B}^\dagger_{\rm in}(t) - \hat{a}^\dagger(t) \otimes \hat{B}_{\rm in}(t)]\right].$$

(c) Using $d\hat{B}_{\rm in}(t)\,d\hat{B}^\dagger_{\rm in}(t) = (N+1)\,dt$, $d\hat{B}_{\rm in}(t)^\dagger\,d\hat{B}(t) = N\,dt$, $d\hat{B}_{\rm in}(t)\,d\hat{B}_{\rm in}(t) = M\,dt$, $d\hat{B}^\dagger_{\rm in}(t)\,d\hat{B}^\dagger_{\rm in}(t) = M^*\,dt$, show that the Taylor expansion of $\hat{U}_I(t+dt,t)$ gives

$$1 + \sqrt{\gamma}[\hat{a}(t) \otimes \hat{B}^\dagger_{\rm in}(t) - \hat{a}^\dagger(t) \otimes \hat{B}_{\rm in}(t)]$$
$$-\tfrac{dt\gamma}{2}[N\hat{a}(t)\hat{a}^\dagger(t) + (N+1)\hat{a}^\dagger(t)\hat{a}(t) - M\hat{a}^\dagger(t)\hat{a}^\dagger(t) - M^*\hat{a}(t)\hat{a}(t)] \otimes \hat{I}_B + O(\gamma^{3/2})$$

(d) The stochastic evolution equation for a cavity operator $\hat{s}$ is

$$\hat{s}(t) = \hat{U}(t+dt,t)^\dagger \hat{s}(t)\hat{U}(t+dt,t) - \hat{s}(t).$$

Using the expansion for $\hat{U}(t+dt,t)$ above, one can show that a cavity operator $\hat{s}$ satisfies the following explicit stochastic equation

$$d\hat{s} = \sqrt{\gamma}[\hat{a}^\dagger \otimes d\hat{B}_{\rm in} - \hat{a} \otimes d\hat{B}^\dagger_{in}, \hat{s}] + \tfrac{\gamma\,dt}{2}\{(N+1)(2\hat{a}^\dagger\hat{s}\hat{a} - \hat{s}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{s})$$
$$+N(2\hat{a}\hat{s}\hat{a}^\dagger - \hat{s}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{s}) + M[\hat{a}^\dagger,[\hat{a}^\dagger,\hat{s}]] + M^*[\hat{a},[\hat{a},\hat{s}]]\} \otimes \hat{I}_B$$

where the time dependence of the operators has been omitted for clarity. Using the result above (which you do *not* need to prove) and the commutation relation $[\hat{a},\hat{a}^\dagger] = 1$, show that for $\hat{s} = \hat{a}$, we obtain the implicit state equation

$$\dot{\hat{a}}(t) = -\tfrac{\gamma}{2}\hat{a}(t) - \sqrt{\gamma}\hat{b}_{in}(t).$$

(e) One can furthermore derive the input-output relation $\hat{b}_{\rm out}(t) = \sqrt{\gamma}\hat{a}(t) + \hat{b}_{\rm in}(t)$. Using this result (which you do *not* need to prove) together with the implicit state equation for the cavity operator $\hat{a}(t)$ above, show that the transfer function of the cavity is

$$\tilde{b}_{out}(s) = \tfrac{s-\gamma/2}{s+\gamma/2}\tilde{b}_{\rm in}(s)$$

where $\tilde{a}(s)$ and $\tilde{b}(s)$ are the Laplace transforms of $\hat{a}(t)$ and $\hat{b}(t)$.

(f) State the Nyquist stability condition and explain if the open-loop cavity system satisfies it.

END OF PAPER