UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2007 1.30 to 4.30

PAPER 6

NOETHERIAN ALGEBRAS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove the non-commutative version of Hilbert's Basis Theorem.

Define a poly-(cyclic or finite) group. Stating clearly any results that you use in addition to the basis theorem, show that the group algebra $\mathbb{Z}G$ is Noetherian whenever G is a poly-(cyclic or finite) group.

2 Let R be a commutative ring and S a multiplicatively closed subset of R. Define the localisation R_S of R at S using its universal property and sketch the proof that it exists.

Explain what it means to localise R at a prime ideal P of R. Show that R_P, the localisation of R at P is always a local ring.

Suppose that R_P contains no non-trivial nilpotent element for each prime ideal P of R. Show that R contains no non-trivial nilpotent elements.

Suppose now that R_P is an integral domain for each prime ideal P. Must R be an integral domain? Justify your answer.

3 Show, using Zorn's Lemma, that every ring R has a simple left R-module.

Define the Jacobson radical $J(R)$ of a ring R. Show that $J(R)$ consists of all elements x in R such that $1 - axb$ is a unit in R for every a and b in R.

Suppose now that R is commutative and I is an ideal in R. Show that the set $S = 1 + I$ is multiplicatively closed and that the localisation I_S of I is contained in $J(R_S)$.

4 State and prove the Artin–Wedderburn Theorem.

Deduce that if G is a finite group and $S_1, \ldots, S_k$ are all the simple $\mathbb{C}G$-modules up to isomorphism then

$$\sum_{i=1}^{k} (\dim_{\mathbb{C}} S_i)^2 = |G|.$$

You may assume that the Jacobson radical of $\mathbb{C}G$ is 0.

5 Define an almost commutative $\mathbb{C}$-algebra. Show that if R is an almost commutative $\mathbb{C}$-algebra then R is Noetherian. Deduce that if $\mathfrak{g}$ is a finite dimensional $\mathbb{C}$-Lie algebra then the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is Noetherian.

6 Let R be a left Noetherian ring and M a finitely generated left R-module. Show that M has a projective resolution consisting of finitely generated projective modules.

Deduce that if M has projective dimension $n < \infty$ then $\mathrm{Ext}^n_R(M, R) \neq 0$.

Find a ring R and an R-module M such that M does not have finite projective dimension as an R-module.

END OF PAPER