UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2007 9.00 to 11.00

PAPER 59

QUANTUM INFORMATION, ENTANGLEMENT AND NONLOCALITY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A quantum measurement is represented by a collection of operators A_i. Write down the completeness equation which these operators must satisfy. Write down an expression for the probability of outcome i resulting from the measurement on a state $|\psi\rangle$. Write down an expression for the state after a measurement which results in outcome i.

Suppose now that quantum systems S_1 and S_2 are in an entangled pure state, and that the measurement is scheduled to be carried out on system S_1 at some fixed time. Show, justifying your answer carefully, that an experimenter who has access only to S_2 has no way of obtaining any information as to whether or not the measurement actually was carried out. (You may quote any principle of quantum mechanics without proof, but may *not* appeal directly to the no-superluminal-signalling postulate.)

2 The Greenberger-Horne-Zeilinger state takes the form

$$|\psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}(\,|\uparrow\rangle_1|\uparrow\rangle_2|\uparrow\rangle_3 - |\downarrow\rangle_1|\downarrow\rangle_2|\downarrow\rangle_3\,)\,.$$

Here $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$ are the eigenstates of σ_z^i, the operator defining the action of σ_z on the i-th particle.

Show that the operators $\sigma_x^1\sigma_y^2\sigma_y^3$, $\sigma_y^1\sigma_x^2\sigma_y^3$, $\sigma_y^1\sigma_y^2\sigma_x^3$ and $\sigma_x^1\sigma_x^2\sigma_x^3$ commute and that $|\psi\rangle_{GHZ}$ is an eigenstate of each of them. By considering the action of these operators on $|\psi\rangle_{GHZ}$, show that the predictions of quantum mechanics and of the Einstein-Podolsky-Rosen criterion are in direct contradiction for a GHZ state of three spacelike separated particles.

3 Let $|0\rangle, |1\rangle, \ldots, |n-1\rangle$ be an orthonormal basis of an n-dimensional Hilbert space, let ω be a complex n-th root of unity, and define $|r\rangle = |r \pmod n\rangle$ for $r \geqslant n$. Show that the states

$$|\psi_{rs}\rangle = \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\omega^{jr}|j\rangle|j+s\rangle$$

are orthonormal.

By using these states, or otherwise, describe a protocol for teleporting an unknown state in a n dimensional Hilbert space.

4 Alice and Bob possess bit strings $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ respectively; Alice's bit string is known to her but not to Bob, and vice versa. They wish to calculate one bit determined by a Boolean function $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ of their joint data; the form of f is known to them both. They possess an unlimited supply of "black boxes" which accept one bit inputs x and y from Alice and Bob respectively, and generate one bit outputs a and b for Alice and Bob respectively, with the property that

$$a + b = x \cdot y \ ,$$

where both sides of the equation are taken modulo 2. The boxes' outputs (a, b) are generated independently, randomly and equiprobably from the two pairs satisfying this constraint. Show how Alice and Bob can enable Bob to calculate f by combining individual calculations, a series of joint black box invocations, and one bit of classical communication from Alice to Bob. (You may cite without proof any relevant theorem about Boolean functions.)

END OF PAPER