UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   **Part III**

Tuesday 5 June 2007   1.30 to 3.30

PAPER 58

INTRODUCTION TO QUANTUM COMPUTATION

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Positive Operator Valued Measurements (POVMs) are a generalised form of measurement. When acting on a single qubit, they are described by a set of m 2×2 positive matrices $\{E_n\}$, where $1 \leqslant n \leqslant m$, satisfying the condition

$$\sum_{n=1}^{m} E_n = \mathbf{1}_2$$

where $\mathbf{1}_N$ is the $N \times N$ identity matrix. We shall restrict ourselves to the case where the matrices are rank one, i.e. $E_n = \alpha_n |\psi_n\rangle\langle\psi_n|$ with positive constants α_n.

(a) Given that $\sum_{n=1}^{N} |n\rangle\langle n| = \mathbf{1}_N$ for a particular complete orthonormal basis $\{|n\rangle\}$, prove that the identity holds for all complete orthonormal bases on the same space.

(b) Consider the special case of $m = 2$. Plot an arbitrary E_1 on the Bloch Sphere, assuming it to be of unit length. Also depict how E_2 is related to it.

(c) In an N-dimensional space, how many linearly independent vectors are required in order to specify an arbitrary point in the space relative to some predefined centre?

(d) In order to characterise the (possibly mixed) state of a qubit, we choose to perform a series of measurements on many identical (independent) copies of the state. What is the minimum value of m required to entirely characterise this state.

(e) Show that for any two non-orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$, there are no measurements that can perfectly distinguish the states (i.e. it is impossible to find E_1 and E_2 such that $\langle\psi_1|E_1|\psi_1\rangle = \langle\psi_2|E_2|\psi_2\rangle = 1$).

2

(a) Consider an N-qubit Hamiltonian

$$H_G = \Delta(|\psi\rangle\langle\psi| + |w\rangle\langle w|)$$

where $\langle\psi|w\rangle = 1/\sqrt{2^N}$, $|\psi\rangle$ and $|w\rangle$ are properly normalised states, and $\Delta > 0$ is a constant. How does a state $|\phi\rangle$ evolve under the action of H_G if $\langle\phi|\psi\rangle = \langle\phi|w\rangle = 0$?

(b) Find the smallest time $t_0 > 0$ such that

$$e^{-iH_G t_0}|\psi\rangle = e^{i\alpha}|w\rangle$$

up to a global phase factor α.

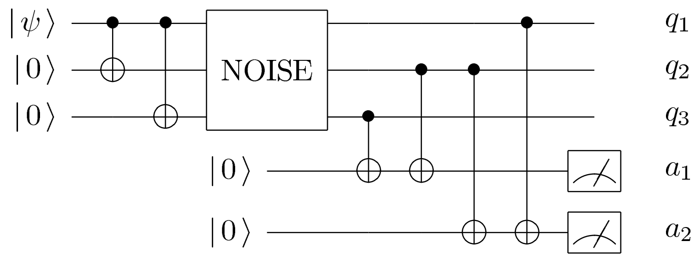
(c) For a given rank one projector, $|\,\phi\rangle\langle\phi\,|$, find a simple expression for

$$U_{|\phi\rangle\langle\phi|} = e^{-i\pi|\phi\rangle\langle\phi|}$$

by, for example, performing a series expansion of the exponential. If $|w\rangle$ is in the computational basis ($w \in \{0,1\}^N$), and $|\psi\rangle = H^{\otimes N}|0\rangle^{\otimes N}$, where H is the Hadamard matrix, how do $U_{|w\rangle\langle w|}$ and $U_{|\psi\rangle\langle\psi|}$ relate to Grover's Search Algorithm?

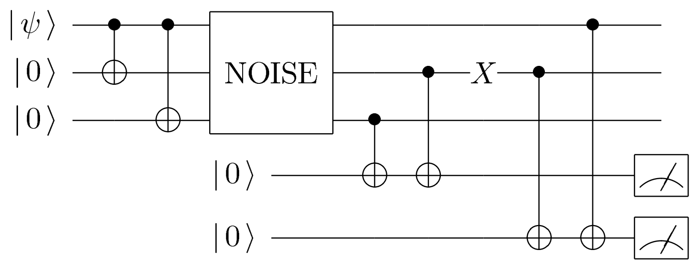
(d) Why might we be able to create a Hamiltonian such as H_G to try to find $|w\rangle$ without already knowing $|w\rangle$?

3 Consider the following circuit. It can be divided into three parts – an encoding step intended to protect the (unknown) quantum state $|\psi\rangle$ against single bit-flip (X) errors, transmission through a noisy channel where a bit-flip occurs independently on each qubit with probability p, and, finally, a circuit that detects any errors and corrects the state, leaving it encoded across the three qubits. The labels q_1, q_2, q_3, a_1 and a_2 are used to refer to the different qubits.



(a) Verify that we can detect a single bit-flip from the noisy channel by measuring the two ancilla qubits (a_1 and a_2) in the computational basis $\{0, 1\}$. Specify what corrections would be required for each possible measurement result.

(b) If no error were to occur during transmission through the noisy channel, but instead a bit-flip error occurs during the error detection circuit as depicted below, how does this error affect the final state after following the process you described in part a?



(c) If bit-flips occur independently on each qubit with probability p due to the noisy channel, and the fault in the error-detection circuit occurs independently with a probability q, give the threshold value for q (in terms of p) below which the sequence of encoding and correcting errors provides an enhancement in robustness over the transmission of a single qubit.

4 In the following, we denote the four Bell states by

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right)$$
$$|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right)$$

(a) Verify, for all pairs of normalized qubit states $|a\rangle$ and $|b\rangle$ where $\langle a|b\rangle = 0$, that

$$\frac{1}{\sqrt{2}}\left(|a\rangle|b\rangle - |b\rangle|a\rangle\right) \;=\; e^{i\alpha}|\Phi_-\rangle$$

up to a global phase factor $e^{i\alpha}$.

(b) Show that all four states $|\Psi_\pm\rangle$ and $|\Phi_\pm\rangle$ are eigenvectors of the operators $X \otimes X$ and $Z \otimes Z$, and give the respective eigenvalues. (X and Z are the standard Pauli matrices.)

(c) Give unitaries U_1, U_2 such that

$$U_1 \otimes U_1|\Psi_+\rangle = |\Psi_-\rangle$$
$$U_2 \otimes U_2|\Psi_+\rangle = |\Phi_+\rangle$$

.

(d) These results directly imply that any two-qubit state can be converted to the state

$$\rho = p\,|\Phi_-\rangle\langle\Phi_-| + \frac{1-p}{3}\left(\,|\Phi_+\rangle\langle\Phi_+| + |\Psi_+\rangle\langle\Psi_+| + |\Psi_-\rangle\langle\Psi_-|\,\right)$$

by the application of local operations, for some $1/4 \leqslant p \leqslant 1$.

When is the partial transpose ($|wx\rangle\langle yz| \rightarrow |wz\rangle\langle yx|$, acting on the computational basis) of ρ not positive? This tells you when the state is entangled.

END OF PAPER