UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 12 June 2007 9.00 to 11.00

PAPER 56

SOLITONS AND INSTANTONS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\phi : \mathbb{R}^{D+1} \longrightarrow \mathbb{R}$ be a smooth function. Define the term *soliton* in the context of scalar field theory with the Lagrangian

$$L = \int_{\mathbb{R}^D} \Big(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}|\nabla \phi|^2 - U(\phi) \Big) d^D \mathbf{x},$$

and use the Derrick scaling arguments to show that solitons can only exist if $D = 1$.

Derive the first order Bogomolny equations and use these equations to find the static kink if $U(\phi) = \phi^6 - 8\phi^4 + 16\phi^2$.

2 Write an essay on sigma model lumps, paying particular attention to the role of topological degree.

3 Consider a Yang–Mills–Higgs field $(A_i, \Phi) : \mathbb{R}^3 \longrightarrow \mathbf{su}(2)$. Define the covariant derivative D_i and the curvature F_{ij} of the potential A_i.

Consider the Bogomolny equations

$$\frac{1}{2}\varepsilon_{ijk} F_{jk} = D_i \Phi \qquad (1)$$

on $\mathbb{R}^3$ and discuss their gauge invariance. State the boundary condition needed to interpret (A_i, Φ) as a non-abelian magnetic monopole.

Show that the one–form

$$\mathcal{A} = A_i dx^i + \Phi d\tau$$

satisfies the anti–self–dual Yang–Mills (ASDYM) equations on $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, where (A_i, Φ) is a solution to (1) which does not depend on $\tau \in \mathbb{R}$. Calculate the four–dimensional Euclidean action density in terms of the three–dimensional field F_{jk}.

Define the term *instanton* in the context of a pure Yang–Mills theory in $\mathbb{R}^4$, and show that $\mathcal{A}$ is not an instanton. What can you deduce about translational symmetries of ASDYM instantons?

4 Define the holomorphic line bundles $\mathcal{O}(n) \longrightarrow \mathbb{CP}^1$ and show that if $n > 0$ the space of holomorphic sections is $H^0(\mathbb{CP}^1, \mathcal{O}(n)) = \mathbb{C}^{n+1}$.

Define $H^1(\mathbb{CP}^1, \mathcal{O}(n))$ with respect of an open cover of your choice, and show that

$$\dim\, H^1(\mathbb{CP}^1, \mathcal{O}(n)) = 0, \qquad \text{if} \quad n > -2.$$

Consider the vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow \mathbb{CP}^1$ and characterise those holomorphic sections of this bundle which vanish at a single point of $\mathbb{CP}^1$.

END OF PAPER