UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 7 June 2007 9.00 to 12.00

PAPER 54

STRING THEORY

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

Minor errors in numerical factors will not be heavily penalized.

The covariant world-sheet action for the bosonic string in flat space-time is

$$I = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \, \sqrt{-\det\gamma}\, \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \, .$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Obtain the general solution of the classical equations of motion for a string moving in 26-dimensional space-time with one circular dimension of periodicity $2\pi R$ in terms of a sum over an infinite set of modes.

Determine the energy-momentum tensor, $\theta_{\alpha\beta}$, and hence the hamiltonian of this system. Point out where an ambiguity arises in passing from the classical to the quantum theory. Show that the spectrum of closed string states is invariant under the transformation known as 'T-duality'.

What is a Dp-brane? A Dp-brane is immersed in the space-time considered in the first part of the question and is oriented so that one of the directions in its world-volume is the circular direction. What is the T-dual description?

Now consider the configuration of two parallel Dp-branes separated by a distance L along the circular direction with a string stretching between them. Determine the energy of the ground state of this string.

Describe *qualitatively* how this system of two parallel Dp-branes is described after a T-duality transformation along the circular direction.

2 Describe, without derivation, the classical equations of motion and constraints for a closed bosonic string moving in d-dimensional Minkowski space-time.

How are physical states defined in terms of the Virasoro generators? What is a null state?

Show that a state of the form

$$L_{-1}\,|\chi_1\rangle$$

is a null state if $|\chi_1\rangle$ is an excited string state that satisfies $L_m|\chi_1\rangle = 0$ $(m > 0)$ and $L_0|\chi_1\rangle = 0$.

Show also that

$$\left(L_{-2} + \frac{3}{4}L_{-1}^2\right)\,|\chi_2\rangle$$

is null if $L_m|\chi_2\rangle = 0$ $(m > 0)$ and $(L_0 + 1)\,|\chi_2\rangle = 0$.

The level-two states of the open string consist of the linear combination

$$|\phi\rangle = \zeta^{(2)}_{\mu\nu}\,\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\,|0,p\rangle + \zeta^{(1)}_{\mu}\,\alpha^{\mu}_{-1}\,|0,p\rangle\,,$$

subject to the physical state conditions. Show that these conditions restrict the polarization tensors to satisfy

$$\zeta^{(2)\rho}_{\ \rho} = -2\alpha^{\mu}_0\,\zeta^{(1)}_{\mu}\,, \qquad \zeta^{(1)}_{\nu} = -\alpha^{\mu}_0\,\zeta^{(2)}_{\mu\nu}\,,$$

where $\alpha^{\mu}_0 = \sqrt{2\alpha'}\,p^{\mu}$ satisfies $\alpha_0^2 = -2$.

Show that the solution of these equations is

$$\zeta^{(2)}_{\mu\nu} = \frac{1}{20}\,\zeta^{(2)\rho}_{\ \rho}(3\alpha_{0\mu}\,\alpha_{0\nu} + \eta_{\mu\nu}) + \alpha_{0\mu}\,A_{\nu} + \alpha_{0\nu}\,A_{\mu} + c_{\mu\nu}\,,$$

$$\zeta^{(1)}_{\mu} = \frac{1}{4}\,\zeta^{(2)\rho}_{\ \rho}\,\alpha_{0\mu} + 2A_{\mu}\,,$$

where A_μ is a general vector that satisfies $\alpha^{\mu}_0\,A_{\mu} = 0$ and $c_{\mu\nu}$ is a general symmetric traceless tensor that satisfies $\alpha^{\mu}_0\,c_{\mu\nu} = 0$.

Hence, show that $|\phi\rangle$ has the form

$$|\phi\rangle = c_{\mu\nu}\,\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\,|o,p\rangle + |n_1\rangle + |n_2\rangle\,,$$

where $|n_1\rangle$ and $n_2\rangle$ are null states.

3 Write down the two-dimensional action (with euclidean signature) that describes the bosonic string in a background with arbitrary space-time metric, $G_{\mu\nu}(X)$, antisymmetric potential, $B_{\mu\nu}(X)$, and scalar field, $\Phi(X)$).

What are the local symmetries of the various terms in this action?

Explain how the Φ term in the action is interpreted in string perturbation theory when Φ is constant.

Describe *in outline* how quantization of the two-dimensional world-sheet theory leads to equations for the background space-time fields, expanded as a series in α'. [Concentrate on conceptual ideas rather than mathematical details.]

Explain the rôle of the Φ term, which is not invariant under Weyl transformations, in ensuring consistency of the theory in non-critical space-time dimensions.

4 The action for the open fermionic string propagating in 10-dimensional Minkowski space-time can be written in the form (in conformal gauge on a lorentzian world-sheet and with $\alpha' = 1/2$)

$$I = \frac{1}{\pi} \int d\sigma d\tau \; (2\partial_+ X^\mu \partial_- X_\mu + i\psi_2^\mu \partial_+ \psi_{2\mu} + i\psi_1^\mu \partial_- \psi_{1\mu}) \; ,$$

where $\psi_a^\mu(\sigma, \tau)$ $(\mu = 0, 1, \ldots, 9)$ are 10 Majorana world-sheet spinors with spinor index $a = 1, 2$.

By considering the surface term that arises in deriving the fermion equations of motion, or otherwise, derive the two possible types of open-string boundary conditions on the fermionic fields.

Assuming the coordinates X^μ satisfy Neumann boundary conditions at both end-points, the supercurrent has modes in the R sector that are defined by by the integer-moded operators

$$F_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} : \alpha_m \cdot \psi_{n-m} :$$

Show that

$$\{F_0, F_0\} = 2L_0 \, .$$

What are the constraints satisfied by physical states of the open fermionic string? Show how these constraints can be solved in the light-cone gauge ($\alpha_n^+ = 0$, $\psi_n^+ = 0$, $n \neq 0$) in the critical dimension ($d = 10$), to give expressions for α_n^- and ψ_n^- as sums of terms quadratic in the transverse modes α_n^i and ψ_n^i (where $i = 1, \ldots, d-2$).

The mass of a state in the NS sector is given by

$$\alpha' \, (\text{mass})^2 = \mathbf{N} - a \, ,$$

where $\mathbf{N} = \sum_1^\infty (\alpha_n^{i\dagger} \cdot \alpha_n^i + \psi^{i\dagger}_{n+\frac{1}{2}} \psi^i_{n+\frac{1}{2}})$. Give one argument that the normal ordering constant takes the value $a = \frac{1}{2}$.

Find the mass of the ground state and the first excited physical state of the open string in the NS (half-integer moded) sector. Describe the ground state in the R (integer-moded) sector. Define the GSO projection operator and outline how this leads to an equal number of bosonic and fermionic states.

[*You may assume the (anti)commutation relations*

$$[\alpha_m^\mu, \alpha_n^\nu] = m \, \delta_{m+n,0} \, \eta^{\mu\nu} \, , \qquad \{\psi_m^\mu, \psi_n^\nu\} = \delta_{m+n,0} \, \eta^{\mu\nu} \quad]$$

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