UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2007 9.00 to 12.00

PAPER 50

QUANTUM FIELD THEORY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The Dirac equation is

$$(i\,\gamma^\mu \partial_\mu - m)\,\psi \;=\; 0$$

where the gamma matrices are given in the chiral representation by,

$$\gamma^0 \;=\; \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}\;, \qquad \gamma^i \;=\; \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Here σ^i are the Pauli matrices and 1_2 is the unit 2×2 matrix.

a) Show that these matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} \;=\; 2\,\eta^{\mu\nu}\,1_4$$

where $\eta^{\mu\nu}$ is the Minkowski metric.

b) Show that the each component of the spinor $\psi(x)$ satisfies the Klein-Gordon equation.

c) Consider the ansatz for plane-wave solutions,

$$\psi(x) \;=\; u\,(\vec{p})\,e^{-ip\cdot x}$$

where $p^2 \;=\; m^2$. Show that this ansatz solves the Dirac equation when

$$u\,(\vec{p}) \;=\; \begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi \\ \sqrt{p\cdot\bar{\sigma}}\;\xi \end{pmatrix}$$

for any 2-component spinor ξ, with $\sigma^\mu \;=\; (1_2, \sigma^i)$ and $\bar{\sigma}^\mu \;=\; (1_2, -\sigma^i)$. Write down the ansatz for negative frequency solutions and solve it.

d) The action of a rotation $\vec{\varphi}$ on the Dirac spinor is given by the matrix

$$S\,[\Lambda] \;=\; \begin{pmatrix} e^{i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix}$$

Write down the spinor $u(\vec{p})$ describing a stationary particle of mass m with

(i) Spin directed up along x^3.

(ii) Spin directed up along x^1.

For each of these cases, write down the spinor corresponding to a massless particle travelling in the positive x^3 direction.

2 **a)** A free real scalar field of mass μ in the Heisenberg picture may be expanded as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a_{\vec{p}}^\dagger\, e^{+ip\cdot x} \right)$$

where $E_{\vec{p}} = \sqrt{\vec{p}\cdot\vec{p} + \mu^2}$ and $a_{\vec{p}}$ and $a_{\vec{p}}{}^\dagger$ satisfy the commutation relations

$$[\, a_{\vec{p}}, a_{\vec{q}} \,] = [\, a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger \,] = 0$$

and

$$[\, a_{\vec{p}}, a_{\vec{q}}^\dagger \,] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})\,.$$

Define the vacuum state $|\,0\,\rangle$. Show that the propagator $\langle\, 0|\phi(x)\,\phi(y)|0\,\rangle$ is given by

$$\langle\, 0\,|\,\phi(x)\,\phi(y)\,|\,0\,\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}\, e^{-ip\cdot(x-y)}\,.$$

b) The Feynman propagator for a real scalar field is defined to be

$$\Delta_F(x-y) = \langle\, 0\,|\,T\phi(x)\,\phi(y)\,|\,0\,\rangle$$

where T stands for time ordering. Show that the propagator may be written as

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot(x-y)}}{p^2 - \mu^2 + i\epsilon}\,.$$

c) The Lagrangian for a real scalar field ϕ interacting with a Dirac spinor ψ is given by

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,\mu^2\phi^2 + \bar{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi - \lambda\phi\,\bar{\psi}\,\psi$$

Draw the lowest order Feynman diagrams for $\psi\psi \to \psi\psi$ scattering and $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering. In both cases, label the incoming particles with 4-momenta p and q, and label the outgoing particles with 4-momenta p' and q'.

d) Write down the amplitude for $\psi\psi \to \psi\psi$ scattering at order λ^2, quoting any Feynman rules that you use.

[*Useful Information:* The Feynman rules state that to each incoming fermion with momentum p and spin r, you should associate the spinor $u^r(\vec{p})$. For outgoing fermions, associate $\bar{u}^r(\vec{p})$. To each incoming anti-fermion with momentum p and spin r, associate a spinor $\bar{v}^r(\vec{p})$. For outgoing anti-fermions, associate $v^r(\vec{p})$.]

3 The Lagrangian for a scalar field φ of mass m and charge e, interacting with the electromagnetic field is

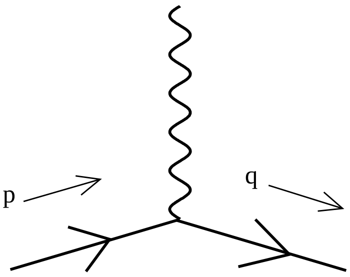
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu \varphi^\star \mathcal{D}^\mu \varphi - m^2 |\varphi|^2$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\mathcal{D}_\mu\varphi = \partial_\mu\varphi + i\, eA_\mu\varphi$.

a Show that this Lagrangian has a gauge symmetry.

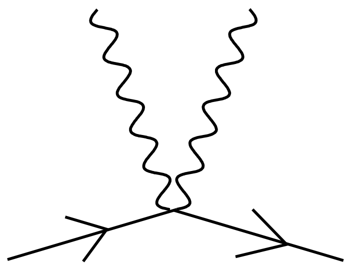
b What is the physical difference between gauge symmetries and global symmetries? Justify your answer.

c The theory contains two interaction vertices with Feynman rules given by



$-i\, e\, (p+q)_\mu$

and



$+\, 2i\, e^2\, \eta_{\mu\,\nu}$

where $\eta_{\mu\nu}$ is the Minkowski metric. Identify the interaction terms in the Lagrangian corresponding to these two vertices.

d) When quantizing the theory in Coulomb gauge $\nabla \cdot \vec{A} = 0$, the naive photon propagator is

$$D_{\mu\nu}(p) = \begin{cases} \dfrac{i}{p^2 + i\,\epsilon}\left(\delta_{ij} - \dfrac{p_i p_j}{|\vec{p}\,|^2}\right) & \mu = i \neq 0\,,\ \nu = j \neq 0 \\ \dfrac{i}{|\vec{p}\,|^2} & \mu, \nu = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Draw the leading order diagrams for $\varphi\bar{\varphi} \to \varphi\bar{\varphi}$ scattering and show that, when the external momenta are on-shell, the naive photon propagator may be replaced by the Lorentz invariant propagator

$$D_{\mu\nu}(p) = -i\,\frac{\eta_{\mu\nu}}{p^2}\,.$$

4 Write an essay on the role of anti-particles in quantum field theory. To obtain full credit the essay should include a discussion on the following topics: the difference between a real and a complex scalar field; the difference between a relativistic and non-relativistic field theory; a comparison of Dirac's hole interpretation of anti-particles and the appearance of anti-particles in quantum field theory.

END OF PAPER