UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2007 1.30 to 4.30

PAPER 5

PRO-p GROUPS

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Give two equivalent definitions of a profinite group, defining terms (such as topological group and inverse limit) as required.

Prove the following:

(i) Every open subgroup of a topological group is closed.

(ii) In a compact topological group every open subgroup is of finite index.

(iii) Inverse limits are unique.

Define the Frattini subgroup of a profinite group. How can the Frattini subgroup of a pro-p group be characterised?

Recall, G is *procyclic* if G is profinite and G/N is cyclic for every open normal subgroup N of G. Let G be a pro-p group and suppose G is procyclic. Prove that G can be topologically generated by one element. (Hint: For a contradiction suppose that G has 2 distinct maximal open subgroups.) Give an example of a procyclic pro-p group.

2 Let G be a finite p-group for p an ODD prime and N a subgroup of G. Explain what it means to say that N is *powerfully embedded* in G and that G is *powerful*.

Let G be a powerful finite p-group and let H be a subgroup of G. Let $\mathrm{d}(G)$ denote the minimal number of elements needed to generate G. By induction on $|G|$ (or otherwise) prove that $\mathrm{d}(H) \leqslant \mathrm{d}(G)$. (You may use the following facts (i) if N is powerfully embedded in G then N^p is powerfully embedded in G (ii) the natural map $\theta : G/G^p \to G^p/G^{p^2}$ given by $xG^p \mapsto x^p G^{p^2}$ is an onto homomorphism.)

[*Hint: Consider* $K = H \cap G^p$. *Furthermore, consider the restriction of* θ *to* HG^p/G^p.]

Define the *rank* of a profinite group. Suppose H is an open subgroup of a profinite group P. Prove that H has finite rank if and only if P has finite rank.

3 Let p be an ODD prime and let G be a pro-p group. Define the *lower p-series* of G.

Let G be a finitely generated powerful pro-p group. Denote the lower p-series of G by $P_i(G)$, and for H a subgroup of G let $\mathrm{d}(H)$ denote the minimal number of elements needed to topologically generate H. Stating clearly any results that you use, prove that the following are equivalent:

(i) G is uniform.

(ii) $\mathrm{d}(P_i(G)) = \mathrm{d}(G)$ for all $i \geqslant 1$.

(iii) $\mathrm{d}(H) = \mathrm{d}(G)$ for every powerful open subgroup of G.

Let G be a uniform pro-p group. Define the operation $+_n$ for $n \geqslant 1$. Prove the following results.

(i) The operation $+_n$ is associative.

(ii) If $x, y \in G$ then $x +_n y \equiv y +_n x \bmod P_{n+2}(G)$.

(iii) If $x, y \in G$ then $x +_n y \equiv x +_{n-1} y \bmod P_n(G)$.

Given G a uniform pro-p group define the intrinsic Lie algebra L_G. Now define a powerful Lie algebra L and explain how you define a uniform pro-p group $(L, *)$. What is the connection between the two structures L_G and $(L, *)$?

4 Write an essay describing the proof of the linearity of uniform pro-p groups.

5 Give two definitions of the Nottingham group, one as a group of formal power series and one as a group of field automorphisms.

Prove that the Nottingham group contains every finite p-group as a subgroup.

What does it mean to say that a profinite group is *hereditarily just infinite*? Sketch a proof that the Nottingham group is hereditarily just infinite, stating clearly any results you use and explaining any notation.

Using the following theorem prove that the Nottingham group is not analytic over any pro-p ring R, again state clearly any results you use.

Theorem. Let G be an R-analytic pro-p group. If G is hereditarily just infinite then one of the following holds:

(i) G is linear over either $\mathbb{Z}_p$ or $\mathbb{F}_p[[t]]$;

(ii) there exists $s \geqslant 1$ such that

$$|G : G^{(n)}| \geqslant p^{3^{n-s}d}$$

for all large n where $d = \dim(G)$ and $G^{(n)}$ denotes the derived series (i.e. $G^{(1)} = \overline{[G, G]}$ and $G^{(n)} = \overline{[G^{(n-1)}, G^{(n-1)}]}$ for $n \geqslant 2$).

END OF PAPER