UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2007 9.00 to 11.00

PAPER 45

ACTUARIAL STATISTICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $S = X_1 + \ldots + X_N$ where $X_1, X_2, \ldots$ are independent identically distributed random variables, N takes values in $\{0, 1, 2, \ldots\}$ and is independent of the X_i's. Show that $\mathbb{E}(S) = \mathbb{E}(N)\mathbb{E}(X_1)$ and $\mathrm{var}(S) = \mathbb{E}(N)\ \mathrm{var}(X_1) + \mathrm{var}(N)(\mathbb{E}(X_1))^2$.

Let S_i be the total amount claimed in a year on the i^{th} policy, $i = 1, \ldots, n$, in a portfolio of n independent policies. Suppose that

$$S_i = \begin{cases} 0 & \text{with probability } 1 - q_i, \\ Y_i & \text{with probability } q_i, \end{cases}$$

where $0 < q_i < 1$ and Y_i has distribution function F_i, mean $\mu_i > 0$ and variance σ_i^2. Find the mean and variance of the total amount T claimed on the whole portfolio in a year.

Suppose that $S_1, \ldots S_n$ are approximated by independent random variables $\widetilde{S}_1, \ldots, \ldots, \widetilde{S}_n$ with $\widetilde{S}_i = \sum_{j=1}^{\widetilde{N}_i} \widetilde{Y}_j^{(i)}$, where $\widetilde{N}_i$ has a Poisson distribution with mean q_i and $\widetilde{Y}_1^{(i)}, \widetilde{Y}_2^{(i)}, \ldots$ are independent identically distributed random variables with distribution function F_i, independent of $\widetilde{N}_i$. Derive the distribution of $\widetilde{T} = \widetilde{S}_1 + \ldots + \widetilde{S}_n$, and find $\mathbb{E}(\widetilde{T})$ and $\mathrm{var}(\widetilde{T})$.

Now suppose that T is approximated by a compound binomial random variable $\widetilde{\widetilde{T}}$; that is, the number of claims has a $\mathrm{Bin}(n, p)$ distribution where $p = (q_1 + \ldots + q_n)/n$, and claims have distribution function $F = \sum_{i=1}^{n} \frac{q_i}{np} F_i$. Find the mean and variance of $\widetilde{\widetilde{T}}$.

Comment briefly on the approximations $\widetilde{T}$ and $\widetilde{\widetilde{T}}$ for T.

2 Claims arrive at an insurance company in a Poisson process with rate $\lambda > 0$. Successive claims $X_1, X_2, \ldots$ are independent identically distributed random variables, independent of the arrivals process, with density

$$f_X(x) = \frac{4}{\beta^2} x e^{-2x/\beta}, \quad x > 0, \quad \beta > 0.$$

The premium income rate is $c(> 0)$. State the net profitability condition in terms of c, λ and β. Assuming this condition is satisfied, find the adjustment coefficient R.

Suppose that $\beta = 1$ for the rest of this question.

Under quota share reinsurance with retained proportion α, find the distribution of the amount paid on the i^{th} claim by the direct insurer.

Let $c = (1+\theta)\lambda \mathbb{E}X_1$ where $\theta > 0$ is the original premium loading, and suppose that the direct insurer pays premiums to the reinsurer with loading factor $\theta_R \geqslant \theta$. Assume $\alpha > 1 - (\theta/\theta_R)$. Taking reinsurance into account, show that the direct insurer satisfies the net profitability condition.

With the above reinsurance arrangements, write down the direct insurer's adjustment coefficient R_α. Explain why the direct insurer aims to maximize R_α. If $\theta = \theta_R$, write down R_α and find the optimal value of α.

3 In a classical risk model with positive safety loading, let $S(t)$ be the total amount claimed in $(0, t]$, c be the premium income rate and $f_X(x)$ be the claim size density. Define the probability $\psi(u)$ of ultimate ruin with initial capital $u \geqslant 0$.

Show that

$$\frac{c}{\lambda}\psi'(u) = \psi(u) - \int_0^u f_X(u-x)\psi(x)dx - \int_u^\infty f_X(x)dx.$$

Suppose that $f_X(x) = \alpha e^{-\alpha x}$, $x > 0$. Show that

$$\psi''(u) + \left(\alpha - \frac{\lambda}{c}\right)\psi'(u) = 0,$$

and find $\psi(u)$ (you may assume $\psi(0) = \frac{\lambda}{\alpha c}$).

Let $L(t) = S(t) - ct$ be the aggregate loss at time $t \geqslant 0$ and let $L = \sup_{t \geqslant 0} L(t)$. Find $\mathbb{P}(L \leqslant u)$ when claims have the above exponential distribution.

4 A car insurance company operates a No Claims Discount System with discount levels 1,2 and 3, corresponding to 0%, 100α% and 100β% discount respectively, where $0 < \alpha < \beta < 1$. If no claim is made during a year, the policyholder moves up one level of discount or stays at level 3. Otherwise the policyholder moves down one level of discount, or stays at level 1. Explain briefly how the discount level of a policyholder starting in level 1 may be modelled as a discrete-time Markov chain with transition matrix of form

$$P = \begin{bmatrix} p_1 & 1-p_1 & 0 \\ p_2 & 0 & 1-p_2 \\ 0 & p_3 & 1-p_3 \end{bmatrix}.$$

The probability that a policyholder has an accident in a year is p, $0 < p < 1$, and assume that the probability of two or more accidents in a year is negligible. If an accident occurs, the cost of repair has a lognormal distribution with density

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\log x - \mu)^2}{2\sigma^2}\right\}, \quad x > 0.$$

If an accident occurs, policyholders only claim if the cost of repair is greater than the additional premiums that would have to be paid over the next two years, assuming no further accidents occur.

Find p_1, p_2 and p_3 in terms of the standard normal distribution function Φ. In terms of p_1, p_2 and p_3, find the proportion of policyholders at each discount level in the steady state. If a policyholder in level 1 makes a claim, find in terms of Φ the expected size of that claim.

END OF PAPER