MATHEMATICAL TRIPOS Part III

Friday 8 June 2007 9.00 to 11.00

PAPER 42

OPTIMAL INVESTMENT

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a utility function that is finite, twice-differentiable, strictly increasing and strictly concave on the interval $(0, \infty)$ and such that the Inada conditions hold. Let the conjugate function $V : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be

$$V(y) = \sup_{x>0}[U(x) - xy].$$

Show that V is finite, twice-differentiable, strictly decreasing and strictly convex on $(0, \infty)$ and satisfies

$$\lim_{y \downarrow 0} V'(y) = -\infty \quad \text{and} \quad \lim_{y \uparrow \infty} V'(y) = 0.$$

Now consider a market with cash (that is, zero-interest rate) and d assets whose prices are given by the d-dimensional process $(S_n)_{n \geqslant 0}$. Assume this market is free of arbitrage. Let

$$u(x) = \sup_{\pi} \mathbb{E}\big[U\big(X_N^{\pi}\big)\big]$$

where X_N^{π} is the wealth at time N for an investor using trading strategy $\pi = (\pi_n)_{n=0}^{N-1}$ with initial wealth $X_0 = x$, and let

$$v(y) = \inf_{Z_N} \mathbb{E}[V(yZ_N)]$$

where the infimum is taken over all state price densities Z_N.

Prove that the inequality

$$u(x) \leq \inf_{y>0}[v(y) + xy]$$

holds for all $x > 0$.

What does it mean to say the market is complete? Prove that if the market is complete then there exists a unique state price density. Compute $u(x)$ for $x > 0$ as explicitly as you can in the case when the market is complete and

$$U(x) = \begin{cases} \log(x) & \text{if } x > 0 \\ -\infty & \text{if } x \leq 0. \end{cases}$$

2 Consider an investor whose wealth $(X_t)_{t\geq 0}$ is given by

$$dX_t = \theta_t \cdot (\mu dt + \sigma dW_t) - C_t dt$$

for constant vector $\mu \in \mathbb{R}^d$ and $d \times d$ matrix σ and a d-dimensional Brownian motion $(W_t)_{t\geq 0}$. Write down the Hamilton-Jacobi-Bellman equation associated with the problem of maximizing

$$\mathbb{E}\left(U_{\text{wealth}}(X_T) + \int_0^T U_{\text{consumption}}(C_s) ds \right)$$

over admissible controls $(\theta_t)_{t\in[0,T]}$ and $(C_t)_{t\in[0,T]}$, where the utility functions U_{wealth} and $U_{\text{consumption}}$ are positive, increasing, and concave on the interval $(0,\infty)$.

Let $V : \mathbb{R}_+ \times [0,T] \to \mathbb{R}_+$ be the solution to the Hamilton-Jacobi-Bellman equation. Prove that

$$\mathbb{E}\left(U_{\text{wealth}}(X_T) + \int_0^T U_{\text{consumption}}(C_s) ds \right) \leq V(X_0, 0).$$

Show that the Hamilton-Jacobi-Bellman equation has a solution of the form $V(x,t) = f(x)g(t)$ in the case $U_{\text{wealth}}(x) = U_{\text{consumption}}(x) = 2\sqrt{x}$.

3 Let $(W_t)_{t\geq 0}$ be a d-dimensional Brownian motion and $\lambda \sim N(\lambda_0, V_0)$ be an independent Gaussian random vector with given mean $\lambda_0 \in \mathbb{R}^d$ and covariance matrix V_0. Let

$$Y_t = \lambda t + W_t$$

and $(\mathcal{G}_t)_{t\geq 0}$ be the filtration generated by $(Y_t)_{t\geq 0}$.

Prove that the conditional law of λ given $\mathcal{G}_t$ is $N(\lambda_t, V_t)$ for parameters λ_t and V_t to be determined.

Show that the process $(\hat{W}_t)_{t\geq 0}$ is a Wiener process adapted to $(\mathcal{G}_t)_{t\geq 0}$ where

$$\hat{W}_t = W_t + \int_0^t (\lambda - \lambda_s) ds.$$

Let

$$Z_t = \det(I + tV_0)^{\frac{1}{2}} e^{-\frac{1}{2}\lambda_t \cdot V_t^{-1}\lambda_t + \frac{1}{2}\lambda_0 \cdot V_0^{-1}\lambda_0}.$$

Prove that $(Z_t)_{t\geq 0}$ is a supermartingale for $(\mathcal{G}_t)_{t\geq 0}$.

4 Consider a market with cash and d assets whose prices have stochastic dynamics

$$dS_t = \mathrm{diag}(S_t)(\mu_t dt + \sigma_t dW_t)$$

for a $\mathbb{R}^d$-valued Wiener process $(W_t)_{t\geq 0}$, a bounded $\mathbb{R}^d$-valued process $(\mu_t)_{t\geq 0}$, and a uniformly elliptic $d \times d$ matrix-valued process $(\sigma_t)_{t\geq 0}$, all adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Consider an investor who does not consume. What is an admissible trading strategy for this investor? What is an arbitrage? Prove that this market is free of arbitrage.

Let

$$Z_t = e^{-\frac{1}{2}\int_0^t |\lambda_s|^2 ds - \int_0^t \lambda_s \cdot dW_s}$$

where $\lambda_t = \sigma_t^{-1}\mu_t$. Prove that the process $(Z_t S_t)_{t\geq 0}$ is a local martingale. Prove that $(Z_t S_t)_{t\geq 0}$ is a true martingale if $(\sigma_t)_{t\geq 0}$ is bounded.

END OF PAPER