UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2007 9.00 to 12.00

PAPER 40

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Use the simplex algorithm to solve the problem

$$\text{maximize}\, x_1 + x_2 \text{ subject to}$$

$$\begin{aligned} x_1 - x_2 &\geqslant -2 \\ x_1 + 2\,x_2 &\leqslant 8 \\ 2\,x_1 + x_2 &\leqslant 8 \\ x_1,\ x_2 &\geqslant 0 \end{aligned}$$

Suppose we now add to this problem the constraint that x_1 and x_2 must be integers. Use Gomory's cutting plane method and the dual simplex algorithm to find all the optimal solutions. Carefully explain the rationale for any new constraints that you introduce.

2

Discuss the representation of the travelling salesman problem as an integer linear programming problem. Show that if there are n cities and all intercity distances are integers that are at most 2^n, then the travelling salesman problem can be expressed as an integer linear program with size of $O(n^3)$ bits.

Define the class of decision problems $\mathcal{NP}$-complete.

Show that if travelling salesman decision problems are $\mathcal{NP}$-complete then integer linear programming decision problems are also $\mathcal{NP}$-complete.

3

Consider the problem '*Is the polyhedron* $P = \{x \in \mathbb{R}^n : Ax \geqslant b\}$ *nonempty?*' Suppose it is known that if P is nonempty then it is contained within the ellipsoid $E = E(z, D) = \{x : (x - z)^\top D^{-1}(x - z) \leqslant 1\}$. Show that if $z \notin P$ and P is non-empty then P must be contained in the intersection of E and some half space H.

Suppose D is the $n \times n$ identity matrix, $H = \{x : x_1 \geqslant 0\}$, and $e_1 = (1, 0, \ldots, 0)\}$. Given that

$$E' = E\left(\frac{e_1}{n+1}, \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1}\; e_1 e_1^\top\right)\right)$$

is an ellipsoid containing $E \cap H$, show that the volume of E' is less than $e^{-1/(2(n+1))}$ times the volume of E. You may use the fact that the volume of $E\,(z, D)$ is proportional to $\sqrt{\det(D)}$.

Briefly discuss the importance of the above result in constructing a polynomial time algorithm for linear programming.

4

Explain what is meant by the characteristic function of a coalitional game. Why is the characteristic function always a superadditive function?

A company is bankrupt and owes three creditors amounts of money $c_1 = 4$, $c_2 = 6$, $c_3 = 9$ (in 10000 s of pounds). Unfortunately it has assets of only $a = 15$. A liquidator proposes to divide these assets so that creditors 1, 2 and 3 receive $x_1 = 8/3$, $x_2 = 14/3$ and $x_3 = 23/3$, respectively.

For the characteristic function defined by

$$v(S) = \max\left\{0,\ \ a - \sum_{i \notin S} c_i\right\}, \qquad S \subseteq \{1,2,3\},$$

and $x = (x_1, x_2, x_3)$, determine which of the following are true. Your answer should include definitions of 'Pareto optimal', 'Shapley value', 'nucleolus' and 'core'.

(a) x is Pareto optimal.

(b) x_1, x_2, x_3 are Shapley values.

(c) x is the nucleolus.

(d) x is in the core.

5

A seller is preparing to sell a used car. He knows that there are just 2 potential buyers. He considers 3 methods of selling.

(a) He offers the car at price p and waits to see if anyone buys it.

(b) He conducts an oral ascending price auction, selling the car to the highest bidder, who must pay his bid.

(c) He modifies (b) by accepting no bid less than his reserve price r . A buyer can win only if he is the highest bidder and bids more than r . The winner (if any) pays his bid.

Suppose that the buyers have independent private valuations of v_1 and v_2 , where *a priori* these can be modelled as independent uniform random variables on $[0, 1]$ (measuring, perhaps, fractions of 2000 pounds). Assuming that the seller chooses p and r optimally, determine his expected revenue under each selling method. Which is best?

Explain why the fact that the expected revenue differs under (b) and (c) does not contradict the revenue equivalence theorem for SIPV auctions.

6

Given a set of n items, with positive weights $\{w_1, w_2, \ldots, w_n\}$, we wish to find the least y such that the items can be placed in 2 bins and the total weight in each bin is no more than y .

(a) Formulate the problem as an integer linear programming problem.

(b) Define the notion of an ϵ–approximation algorithm for a minimization problem.

Let $s = \sum_i w_i$ and $w_{\max} = \max_i w_i$. Consider an algorithm in which we place the items in the bins in the order $1, 2, \ldots, n$, always placing an item in a bin that presently contains the least total weight. Show that just before it receives item j , the bin which receives item j contains total weight that is no more than $(1/2)(s - w_j)$ (for any j). Deduce that the algorithm achieves $y \leqslant s/2 + w_{\max}/2$. Hence determine the least ϵ for which this algorithm can be claimed to be an ϵ–approximation algorithm.

Hint: consider an example of 3 items, with weights 1, 1 and 2.

(c) Using the bin-packing problem above to illustrate your ideas, discuss other means of obtaining approximate solutions to combinatorial optimization problems.

END OF PAPER