UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2007 9.00 to 12.00

PAPER 4

FINITE DIMENSIONAL LIE ALGEBRAS AND THEIR REPRESENTATIONS

*Attempt **ALL** questions.*

*There are **FIVE** questions in total.*

Question 2 carries the most weight.

All Lie algebras are over $\mathbb{C}$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathfrak{g}$ be the 3-dimensional Lie algebra over $\mathbb{C}$ with basis p, q, c and bracket

$$[p,q] = c \qquad [c,p] = [c,q] = 0.$$

(i) Determine the derived and central series for $\mathfrak{g}$. Is $\mathfrak{g}$ nilpotent, solvable or neither?

(ii) Determine all the irreducible finite dimensional representations of $\mathfrak{g}$.

(iii) Find an irreducible faithful representation of $\mathfrak{g}$.

(iv) Find a faithful finite dimensional representation of $\mathfrak{g}$.

2 Let

$$\mathfrak{g} = \mathfrak{so}_{2n+1} = \left\{A \in Mat_{2n+1} | AJ + JA^T = 0\right\}, \quad J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

and $\mathfrak{t} =$ diagonal matrices in $\mathfrak{g}$.

(i) Decompose $\mathfrak{g}$ as a $\mathfrak{t}$-module, and hence write the roots R for $\mathfrak{g}$. Choose positive roots to be those occurring in upper triangular matrices. Write down the positive roots R^+, the simple roots π, the highest root θ, and the fundamental weights. Write down ρ.

Draw the Dynkin diagram and label it by simple roots. Draw the extended Dynkin diagram.

(ii) Show that $\mathfrak{so}_5 \simeq \mathfrak{sp}_4$.

(iii) For each root $\alpha \in R$, write the reflection $s_\alpha : \mathfrak{t} \to \mathfrak{t}$ explicitly. Describe the Weyl group W (you do not need to prove your answer).

(iv) Let $V = \mathbb{C}^{2n+1}$ be the standard representation of $\mathfrak{so}_{2n+1}$. Draw the crystal of V, and of $V \otimes V$. Write the highest weight of each irreducible summand of $V \otimes V$.

3

(i) State the Weyl dimension formula, briefly defining the notation you use.

(ii) Draw the root system of G_2, and the fundamental weights ω_1, ω_2.

Write down the dimension of the irreducible representation with highest weight $n_1\omega_1 + n_2\omega_2, \quad n_1, n_2 \in \mathbb{N}$.

4 Let $\mathfrak{g}$ be a Lie algebra, V a representation of $\mathfrak{g}$. Show there is a homomorphism of $\mathfrak{g}$-modules from $\mathfrak{g} \otimes V$ to V. Hence conclude that if $\mathfrak{g}$ is semisimple and V is a non-trivial simple representation, that V occurs as a summand of $\mathfrak{g} \otimes V$.

5

(i) Let V be a representation of sl_2. Define the character of V, and explain why it is a unimodal Laurent polynomial.

(ii) Let

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n][n-1]\dots[n-k+1]}{[k][k-1]\dots[1]}, \quad \text{where } [a] = \frac{q^a - q^{-a}}{q - q^{-1}}.$$

Show $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is a unimodal Laurent polynomial.

END OF PAPER