UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2007 9.00 to 12.00

# PAPER 38

# INTERACTING PARTICLE SYSTEMS

*Attempt* ***FOUR*** *questions.*

*There are* ***SIX*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Let $G = (V, E)$ be a finite connected graph with $|E| \geq 1$, considered as an electrical network with strictly positive conductances $w_e$, $e \in E$, and source $s$, sink $t$. State Kirchhoff's first and second laws for the currents and potential differences of the network. State Ohm's law.

Let $X = (X_n : n \geq 0)$ be a Markov chain on the state space $V$ with transition matrix

$$p_{xy} = \frac{w_e}{\sum_{f \sim x} w_f}$$

where $e$ is the edge $\langle x, y \rangle$ and the summation is over all edges $f$ incident to $x$. Thus $p_{xy} = 0$ if either $x = y$ or $x$ is not a neighbour of $y$. The chain starts at $X_0 = s$, and it stops at the first time it visits $t$.

Let $u_{xy}$ be the expected total number of one-step transitions of the chain from $x$ to $y$; each transition from $x$ to $y$ counts $+1$, and from $y$ to $x$ counts $-1$. Show that $u$ satisfies the two Kirchhoff laws with a total flow of one, and deduce that $u_{xy}$ equals the current along $\langle x, y \rangle$ from $x$ to $y$ when the total flow equals one.

[*A clear statement should be given of any general result to which you appeal.*]

**2** Let $\Omega = \{0, 1\}^E$ where $E$ is a finite set. Define an *increasing* subset of $\Omega$. For increasing subsets $A$, $B$ of $\Omega$, define the subset $A \circ B$ [sometimes written $A \square B$] containing vectors $\omega \in \Omega$ for which '$A$ and $B$ occur disjointly'.

State the BK 'disjoint-occurrence' inequality for the product measure $P_p$ on $\Omega$ with density $p$.

Consider bond percolation on $\mathbb{Z}^2$ with density $p$, and let $A$ be the event that there exists an open path that crosses the rectangle $[0, 2n] \times [0, 2n - 1]$ from its left side to its right side. Show that

$$P_p(A) + P_{1-p}(A) = 1.$$

By considering the open clusters at vertices of the form $(n, y)$ for $0 \leq y \leq 2n$, show that

$$P_{\frac{1}{2}}(\mathrm{rad}(C) \geq n) \geq \frac{1}{2\sqrt{n}}$$

where $C$ is the cluster at the origin 0 and $\mathrm{rad}(C) = \max\{n : 0 \leftrightarrow \partial\Lambda_n\}$ with $\Lambda_n = [-n, n]^2$.

**3** Let $G = (V, E)$ be a finite graph. Let $p \in (0, 1)$, $q \in \{2, 3, \dots\}$, and write $\Omega = \{0, 1\}^E$ and $\Sigma = \{1, 2, \dots, q\}^V$. Let $\kappa$ be the probability measure on $\Omega \times \Sigma$ given by

$$\kappa(\omega, \sigma) = \frac{1}{Z} \prod_{e \in E} \Big\{ (1-p)\delta_{\omega(e),0} + p\delta_e(\sigma)\delta_{\omega(e),1} \Big\},$$

where $\delta_e(\sigma) = \delta_{\sigma_x, \sigma_y}$ for $e = \langle x, y \rangle \in E$.

Show that the first marginal measure of $\kappa$ is the random-cluster measure $\phi_{p,q}$, and the second marginal measure is the Potts measure $\pi_{\beta,q}$, where $p = 1 - e^{-\beta}$. Derive the conditional measure on $\Omega$ given the vertex-configuration $\sigma$, and the conditional measure on $\Sigma$ given the edge-configuration $\omega$.

Prove that

$$\left(1 - \frac{1}{q}\right) \phi_{p,q}(x \leftrightarrow y) = \pi_{\beta,q}(\sigma_x = \sigma_y) - \frac{1}{q},$$

and explain how this can be used to relate the phase transitions of the random-cluster and the Potts models.

**4** Let $\Omega = \{0, 1\}^E$ where $E$ is a finite set, and let $\mu_1$ and $\mu_2$ be probability measures on $\Omega$. Explain what is meant by saying that $\mu_1$ dominates $\mu_2$ stochastically. State the Holley condition for this to occur.

Let $G = (V, E)$ be a finite graph, and let $\phi_{p,q}$ be the random-cluster measure on $G$ with parameters $p$ and $q$. Prove that

$$\phi_{p',1} \leq_{\text{st}} \phi_{p,q} \leq_{\text{st}} \phi_{p,1}, \qquad q \geq 1,\ p \in (0, 1),$$

where $p' = p/[p + q(1-p)]$ and $\leq_{\text{st}}$ denotes stochastic ordering. [You may need the fact that $k(\omega) + \eta(\omega)$ is a non-decreasing function of $\omega$, where $k(\omega)$ is the number of open clusters of $\omega$ and $\eta(\omega)$ is the number of open edges.]

Let $p_{\text{c}}(q)$ denote the critical point of the (wired) random-cluster measure on $\mathbb{Z}^d$, where $q \geq 1$. Show that $p_{\text{c}}(1) \leq p_{\text{c}}(q)$, and derive an upper bound for $p_{\text{c}}(q)$ in terms of $p_{\text{c}}(1)$.

**5** Describe the graphical representation of the contact model on $\mathbb{Z}^d$, in terms of Poisson processes of deaths and infection with respective intensities $\delta$ and $\lambda$. Let $\delta = 1$, and write $\theta(\lambda)$ for the probability that an initial infection at the origin persists in $\mathbb{Z}^d$ for all future time. Explain why $\theta$ is a non-decreasing function, and define the critical point $\lambda_c = \lambda_c(d)$ of the process.

Prove that $\lambda_c \geq (2d)^{-1}$.

By coupling the $d$-dimensional process with the one-dimensional process with infection rate $d\lambda$, or otherwise, show that $\lambda_c(d) \leq d^{-1}\lambda_c(1)$.

**6** Let $G = (V, E)$ be a finite graph, and $\Sigma = \{-1, +1\}^V$. For $x \in V$, $\sigma \in \Sigma$, let $N(x) = \Sigma_{y \sim x}\sigma_y$ be the sum of the states of the neighbours of $x$. Consider a discrete-time Markov chain $X = (X_n : n \geq 0)$ on $\Sigma$ with transition probabilities

$$p(\sigma_x, \sigma^x) = \frac{1}{|V|} \cdot \frac{e^{2N(x)}}{1 + e^{2N(x)}},$$
$$p(\sigma^x, \sigma_x) = \frac{1}{|V|} \cdot \frac{1}{1 + e^{2N(x)}},$$

for $x \in V$, $\sigma \in \Sigma$, where $\sigma_x$ (respectively, $\sigma^x$) is the configuration obtained from $\sigma$ by setting the value $-1$ (respectively, $+1$) at the position labelled $x$. Let $p(\sigma, \sigma') = 0$ if $\sigma, \sigma'$ differ at more than one vertex. Show that $X$ is a (time-)reversible Markov chain with respect to the Ising measure

$$\pi(\sigma) = \tfrac{1}{Z} \exp\left(\sum_{x \sim y} \sigma_x \sigma_y\right), \quad \sigma \in \Sigma,$$

where the summation is over all unordered pairs of neighbouring vertices.

Explain how the chain $X$ may be used in a system of 'coupling from the past' in order to generate a random sample with measure $\pi$. Prove that the 'coupling from the past' algorithm terminates in finite time, with probability one.

## END OF PAPER