UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2007 1.30 to 3.30

PAPER 37

SPREAD OF EPIDEMICS AND RUMOURS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A population is closed and consists of n individuals that can be either susceptible, infected or removed. A susceptible is infected at a rate λ times the proportion of infected individuals. Once infected an individual stays infectious for a random length of time following an exponential distribution with parameter γ and is then removed. After removal, an individual loses its immunity and becomes susceptible again at rate ν.

(*a*) Give the transition rates of the Markov process describing the evolution of the epidemic.

(*b*) Derive a differential equation that approximates the dynamics of the epidemics.

Justify carefully all the steps of the proof. You may use, without proofs, Gronwall's lemma and the following result:

For a standard Poisson process $(N(t) : t \geqslant 0)$, *and for positive* T *and* ϵ, *we have*

$$\mathbb{P}\Big(\sup_{t\in[0,T]} |N(t) - t| \geqslant \epsilon T\Big) \leqslant 2e^{-Th(\epsilon)} ,$$

where $h(x) = (1 + x)\log(1 + x) - x$.

2 We consider an Erdös–Renyi random graph $G(n, p)$ with n nodes, where each pair of nodes is connected independently with probability p. If $k \leqslant n$ and $i_1, i_2, \ldots, i_k$ are distinct nodes in $G(n, p)$, we say that the graph contains the k-clique $(i_1, i_2, \ldots, i_k)$ if all the links (i_l, i_m), (l and m distinct in $\{1, 2, \ldots, k\}$) are present in $G(n, p)$. Let X be the number of k-cliques in $G(n, p)$.

(*a*) Prove that if

$$\lim_{n\to\infty} \binom{n}{k} p^{k(k-1)/2} = 0 ,$$

then the number of k-cliques in $G(n, p)$ is zero with probability tending to 1.

(*b*) Assume that

$$\lim_{n\to\infty} \binom{n}{k} p^{k(k-1)/2} = \infty .$$

Show that

$$\mathbb{E}(X^2) = \sum_{l=0}^{k} \binom{n}{k}\binom{k}{l}\binom{n-k}{k-l} p^{2\binom{k}{2}-\binom{l}{2}} .$$

Conclude that

$$\lim_{n\to\infty} \mathbb{P}(\text{there is at least one } k\text{-clique in } G(n, p)) = 1 .$$

3 Let $G(n,p)$ denote the Erdös–Renyi random graph with n nodes, where each pair of nodes is connected independently with probability p. Suppose that

$$\lim_{n\to\infty} \binom{n}{3} p^3 = \alpha \in (0,\infty) .$$

Using the Stein-Chen method, show that the number of 3-cliques (or triangles) X in $G(n,p)$ converges in total variation to a Poisson distribution with parameter α.

What is the probability of having at least one triangle?

4 Let $(M_k)_k$ be a discrete martingale satisfying $|M_k - M_{k-1}| \leqslant c_k$, for all $k \geqslant 1$.

Prove that, for any integer n and positive constant x

$$\mathbb{P}\big(|M_n - M_0| \geqslant x\big) \leqslant 2\exp\left(\frac{-x^2}{\sum_{k=1}^{n} c_k^2}\right) .$$

END OF PAPER