UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2007 1.30 to 4.30

PAPER 35

STOCHASTIC NETWORKS

*Attempt **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the Erlang fixed point approximation for a loss network with fixed routing.

Establish the existence of the fixed point by means of an appeal to the Brouwer fixed point theorem, or otherwise.

Establish the uniqueness of the fixed point, by relating it to a carefully stated optimization problem or otherwise.

Show, by means of an example or otherwise, that in a loss network with alternative routing the Erlang fixed point approximation may not be unique.

2 Write an essay on mathematical models of traffic flow through networks. Your essay should cover the following topics, but need not be restricted to them.

(a) The definition of a Wardrop equilibrium.

(b) The relationship between a Wardrop equilibrium and optimization formulations of network flow.

(c) Braess' paradox.

3 Outline a mathematical model of the slotted infinite-population ALOHA random access protocol, obtaining the recurrence

$$N_{t+1} = N_t + Y_{t-1} - I[Z_t = 1],$$

where $Z_t = 0, 1$ or $*$ according as $0, 1$ or more than 1 packets are transmitted in slot $(t, t+1)$, and Y_t is the number of arrivals in slot $(t, t+1)$. What does N_t represent?

Prove that for any positive arrival rate

$$P\{\exists J < \infty : Z_t = *, \text{ for all } t \geqslant J\} = 1.$$

Discuss whether we can expect a similar result for a finite-population model.

4 Let J be a set of resources, and R a set of routes, where a route $r \in R$ identifies a subset of J. Let C_j be the capacity of resource j, and suppose the number of flows in progress on each route is given by the vector $n = (n_r,\, r \in R)$. A rate allocation $x = (x_r,\, r \in R)$ is *feasible* if

$$x_r \geqslant 0,\ r \in R,\ \text{and} \sum_{r: j \in r} n_r x_r \leqslant C_j, \quad j \in J.$$

A rate allocation is *proportionally fair* if it is feasible and if, for any other feasible rate allocation $y = (y_r,\, r \in R)$,

$$\sum_{r \in R} n_r \left(\frac{y_r - x_r}{x_r} \right) \leqslant 0.$$

Show that a proportionally fair rate allocation solves the optimization problem

$$\begin{aligned} \text{maximize} \quad & \sum_r n_r \log x_r \\ \text{over} \quad & x_r \geqslant 0, \quad r \in R \\ \text{subject to} \quad & \sum_{r: j \in r} n_r x_r \leqslant C_j, \quad j \in J. \end{aligned}$$

Consider a *linear* network with resources $J = \{1, 2, \ldots, I\}$, each of unit capacity, and routes $R = \{0, 1, 2, \ldots, I\}$ where we use the symbol 0 to represent a route $\{1, 2, \ldots, I\}$ which traverses the entire set of resources, and the symbol i to represent a route $\{i\}$ through a single resource, for $i = 1, 2, \ldots, I$. Show that under a proportionally fair rate allocation

$$x_0 n_0 + x_i n_i = 1 \quad \text{if} \quad n_i > 0, \quad i = 1, 2, \ldots, I$$

and

$$x_0 = \frac{1}{n_0 + \sum_{i=1}^{I} n_i} \quad \text{if} \quad n_0 > 0.$$

Suppose now that flows describe the transfer of documents through a network, that new flows originate as independent Poisson processes of rates $\nu_r, r \in R$, and that document sizes are independent and exponentially distributed with parameter μ_r for each route $r \in R$. Determine the transition intensities of the resulting Markov process $n = (n_r,\, r \in R)$. Show that the stationary distribution of the Markov process $n = (n_r,\, r \in R)$ takes the form

$$\pi(n) = B^{-1} \binom{\sum_{r=0}^{I} n_r}{n_0} \prod_{r=0}^{I} \left(\frac{\nu_r}{\mu_r} \right)^{n_r},$$

where B is a normalising constant.

5 Derive Chernoff's bound

$$P\{Y \geqslant 0\} \leqslant \inf_{s \geqslant 0} \mathbb{E}[e^{sY}].$$

Let

$$X = \sum_{j=1}^{J} \sum_{i=1}^{n_j} X_{ji}$$

where X_{ji} are independent random variables with

$$\alpha_j(s) = \frac{1}{s} \log \mathbb{E}[e^{s\, X_{ji}}],$$

for $i = 1, 2, \ldots, n_j$. Show that

$$\sum_{j=1}^{J} n_j \alpha_j(s) \leqslant C - \frac{\gamma}{s} \Rightarrow P\{X \geqslant C\} \leqslant e^{-\gamma},$$

and briefly discuss the interpretation of $\alpha_j(s)$ as an *effective bandwidth*.

In the case where $X_{ji} \sim N(\lambda_j, \sigma_j^2)$, show that the above implication can be written in the form

$$\sum_{j=1}^{J} n_j \lambda_j + \left(2\gamma \sum_{j=1}^{J} n_j \sigma_j^2\right)^{1/2} \leqslant C \Rightarrow P\{X \geqslant C\} \leqslant e^{-\gamma}.$$

Under the same distributional assumptions on X_{ji}, determine a necessary and sufficient condition on $n_1, n_2, \ldots, n_J$ such that $P\{X \geqslant C\} \leqslant e^{-\gamma}$, and show that it can be written in the form

$$\sum_{j=1}^{J} n_j \lambda_j + \phi \left(\sum_{j=1}^{J} n_j \sigma_j^2\right)^{1/2} \leqslant C$$

for a constant ϕ to be determined.

END OF PAPER