UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 7 June 2007 9.00 to 12.00

PAPER 34

QUANTUM INFORMATION THEORY

*Attempt **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Prove that the von Neumann entropy is subadditive, i.e.

$$S(\rho_{AB}) \leqslant S(\rho_A) + S(\rho_B), \tag{1}$$

where ρ_{AB} is the density matrix of a bipartite system AB and ρ_A, ρ_B are the reduced density matrices of the two subsystems A and B respectively.

(b) Using the bound (1) or otherwise, prove the concavity of the von Neumann entropy

$$S\left(\sum_{i=1}^{r} p_i\,\rho_i\right) \geqslant \sum_{i=1}^{r} p_i\,S(\rho_i),$$

where $p_i \geqslant 0$, $\sum_{i=1}^{r} p_i = 1$ and ρ_i , $(i = 1, \ldots, r)$ are density matrices.

(c) Consider a quantum system A which is in a state ρ_i with probability p_i , and let σ be some other density matrix acting on the Hilbert Space $\mathcal{H}_A$ of the system A . Prove that

$$\sum_i p_i\,S(\rho_i\,||\,\sigma) = \sum_i p_i\,S(\rho_i\,||\,\bar{\rho}) + S(\bar{\rho}\,||\,\sigma)\,. \tag{2}$$

In the above, $\bar{\rho} := \sum_i p_i\rho_i$ and the notation $S(\omega\,||\,\sigma)$ denotes the relative entropy of the states ω and σ .

2 Consider a quantum information source defined by a sequence of density matrices $\rho^{(n)}$ acting on Hilbert spaces $\mathcal{H}^{\otimes n}$, and given by

$$\rho^{(n)} = \sum_k p_k^{(n)} |\Psi_k^{(n)}\rangle \langle \Psi_k^{(n)}| , \tag{3}$$

with $p_k^{(n)} \geqslant 0$ and $\sum_k p_k^{(n)} = 1$. Here $\mathcal{H}$ denotes the Hilbert space of a single qubit. Note that the state vectors $|\Psi_k^{(n)}\rangle$ need not be mutually orthogonal.

(a) State a compression–decompression scheme $\mathcal{C}^{(n)}$- $\mathcal{D}^{(n)}$ for such a source and define the corresponding rate of compression. Define the ensemble average fidelity F_n and state a condition under which the compression-decompression scheme is considered to be reliable.

If the density matrix $\rho^{(n)}$ given by (3) satisfies the relation

$$\rho^{(n)} = \pi^{\otimes n} , \tag{4}$$

where π is a density matrix acting in the Hilbert Space $\mathcal{H}$, then the quantum information source is said to be memoryless.

(b) Express the eigenvalues, eigenstates and von Neumann entropy of $\rho^{(n)}$ in terms of the corresponding quantities of the density matrix π.

(c) For any given $\epsilon > 0$, define the ϵ-typical subspace $\mathcal{T}_\epsilon^{(n)}$ of $\rho^{(n)}$ and state the Typical Subspace Theorem.

(d) Define a compression–decompression scheme for such a source, for which the ensemble average fidelity F_n satisfies the bound

$$F_n \geq 2 \sum_k p_k^{(n)} \alpha_k^2 - 1 , \tag{5}$$

where $\alpha_k := ||P_\epsilon^{(n)} |\Psi_k^{(n)}\rangle||$, with $P_\epsilon^{(n)}$ being the orthogonal projection onto $\mathcal{T}_\epsilon^{(n)}$.

(e) Using the above bound (5) and the Typical Subspace Theorem, prove that if $R > S(\pi)$ then there exists a reliable compression scheme of rate R, for the memoryless source given by (4). Here $S(\pi)$ denotes the von Neumann entropy of the state π.

3 The action of the depolarizing channel on the state ρ of a qubit is given by

$$\Phi(\rho) = (1-p)\rho + \frac{p}{3}(\sigma_x \rho_x \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z), \tag{6}$$

where $0 < p < 1$ and σ_x, σ_y and σ_z are the Pauli matrices.

(a) Prove that the depolarizing channel can alternatively be expressed as follows, for some $0 < q < 1$:

$$\Phi(\rho) = (1-q)\rho + q\frac{\mathbf{I}}{2}\,, \tag{7}$$

where $\mathbf{I}$ is the identity operator acting on the single qubit Hilbert space. Hence find the relation between p and q.

(b) Derive the effect of the depolarizing channel on the Bloch sphere, hence justifying its name.

(c) Write an expression for the Holevo χ quantity for an ensemble of quantum states $\mathcal{E} := \{p_i, \rho_i\}$. Express $\chi(\mathcal{E})$ in terms of the relative entropy and prove that it can never increase under a quantum operation.

(d) State the Holevo–Schumacher–Westmoreland (HSW) Theorem and use it to derive the product state capacity of a qubit depolarizing channel with parameter q, defined by (7).

4

(a) Let $\mathcal{H}_A$, $\mathcal{H}_B$ be two Hilbert Spaces, each of dimension d. Write an expression for a maximally entangled state $|\Psi_{AB}\rangle$, of size d, in the Hilbert Space $\mathcal{H}_A \otimes \mathcal{H}_B$ and explain why it is said to be *maximally* entangled.

(b) Prove that any arbitrary vector $|\phi_A\rangle \in \mathcal{H}_A$ can be expressed in terms of the maximally entangled state $|\Psi_{AB}\rangle$, as follows:

$$|\phi_A\rangle = \langle \phi_B^* | \tilde{\Psi}_{AB}\rangle \,, \tag{8}$$

via the relative state method. Here $|\tilde{\Psi}_{AB}\rangle := \sqrt{d}|\Psi_{AB}\rangle$, and $|\phi_B^*\rangle$ is the *index state* in $\mathcal{H}_B$ that yields $|\phi_A\rangle$.

(c) Prove that the pure state resulting from the action of any arbitrary operator M_A on a state vector $|\phi_A\rangle \in \mathcal{H}_A$ can be obtained as a relative state from the state $(M_A \otimes I_B)|\tilde{\Psi}_{AB}\rangle$.

(d) It can be shown that if $\Phi_A : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_A)$ is a linear, completely positive trace–preserving (CPT) map, then

$$\Phi_A(|\phi_A\rangle\langle\phi_A|) = \langle \phi_B^* | (\Phi_A \otimes id_B)(|\tilde{\Psi}_{AB}\rangle\langle\tilde{\Psi}_{AB}|)|\phi_B^*\rangle \,. \tag{9}$$

Using this result, prove that any linear CPT map, Φ_A, can be written in the Kraus form, i.e.,

$$\Phi_A(\rho) = \sum_k A_k \rho A_k^\dagger \,,$$

for any $\rho \in \mathcal{B}(\mathcal{H}_A)$, where the A_k are linear operators in $\mathcal{B}(\mathcal{H}_A)$, satisfying

$$\sum_k A_k^\dagger A_k = \mathbf{I}_A \,,$$

with $\mathbf{I}_A$ being the identity operator in $\mathcal{B}(\mathcal{H}_A)$.

5

(a) State the generalized measurement postulate and state the condition under which it reduces to a projective measurement.

(b) Suppose a projective measurement described by a set of projection operators $\{P_i\}$ is performed on a quantum system, but we never learn the result of the measurement. If the state of the system before the measurement was ρ then the state after the measurement is given by

$$\rho' = \sum_i P_i \rho P_i \,.$$

Prove that the entropy of this final state is at least as great as the original entropy:

$$S(\rho') \geq S(\rho)\,,$$

with equality if and only if $\rho = \rho'$.

(c) Consider a qubit which is in the state ρ with Bloch vector $\vec{s} = (1/3, 1/2, 1/5)$. What is the probability that a projective measurement of the spin of the qubit along the Z–axis will yield a value $+1$?

END OF PAPER