UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2007 1.30 to 4.30

PAPER 33

ADVANCED PROBABILITY

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this problem, for $n \geqslant 0$ we let $D_k^n = [k2^{-n}, (k+1)2^{-n})$, $0 \leqslant k < 2^n$ be the dyadic sub-intervals of $[0, 1)$ with level n . We let $\mathcal{F}_n = \sigma(\{D_k^n\, , \, 0 \leqslant k < 2^n\})$ be the sub-σ-algebra of the Borel σ-algebra $\mathcal{B}([0, 1))$ that is generated by the dyadic subintervals of $[0, 1)$ with level n . The Lebesgue measure on $([0, 1)\, , \mathcal{B}([0, 1)))$ is denoted by λ . The expectations below are understood with respect to the probability space $([0, 1), \mathcal{B}\,([0, 1)), \lambda)$.

a) Let μ be a finite non-negative measure on $([0, 1), \mathcal{B}\,([0, 1)))$. For $n \geqslant 1$, define

$$X_n(x) \;=\; 2^n \sum_{k=0}^{2^n-1} \mu(D_k^n)\, \mathbb{1}_{D_k^n}(x)\,, \qquad x \in [0, 1)\,.$$

Show that $(X_n\, , \; n \geqslant 0)$ is a martingale in some filtered probability space to be made explicit.

b) Justify that $X_n \to X_\infty$ a.s. as $n \to \infty$, where X_∞ is integrable and for every $n \geqslant 0$, $X_n \geqslant E\,[X_\infty|\mathcal{F}_n]$ a.s.

c) We let $X_\infty \cdot \lambda$ be the measure with density X_∞ with respect to λ , meaning that

$$X_\infty \cdot \lambda(A) \;=\; E[X_\infty \mathbb{1}_A]\,, \qquad A \in \mathcal{B}\,([0, 1))\,.$$

(i) Use b) to show that $\mu(f) \geqslant X_\infty \cdot \lambda(f)$ for every non-negative measurable function f, and conclude that $\nu = \mu - X_\infty \cdot \lambda$ defines a non-negative measure on $([0, 1), \mathcal{B}\,([0, 1)))$. If ν_n, λ_n denote the restrictions of ν and λ to $\mathcal{F}_n$, show that ν_n admits a density Y_n with respect to λ_n, which is given by

$$Y_n \;=\; X_n - E\,[X_\infty|\mathcal{F}_n]\,.$$

(ii) Show that $\lim_{n\to\infty} Y_n = 0$ a.s. on the probability space $([0, 1), \mathcal{B}\,([0, 1)), \lambda)$.

(iii) On the other hand, by estimating the ν-measure of the event $\{Y_n \leqslant \varepsilon\}$, show that

$$\nu\left(\left\{x \in [0, 1) : \limsup_{n\to\infty} Y_n(x) \;=\; 0\right\}\right) \;=\; 0\,.$$

2 Let $n \geqslant 1$ and $\theta_1, \ldots, \theta_n > 0$ be positive real numbers with sum

$$S := \sum_{i=1}^{n} \theta_i \leqslant 1 .$$

On some probability space $(\Omega, \mathcal{F}, P)$ consider n independent random variables $U_1, \ldots, U_n$ all uniformly distributed on $[0,1]$, and let $\mathcal{F}_t$ be the σ-algebra generated by the events $\{U_i \leqslant s\}$, $1 \leqslant i \leqslant n$, $0 \leqslant s \leqslant t$. We define

$$X_t = \sum_{i=1}^{n} \theta_i \mathbb{1}_{\{U_i \leqslant t\}}, \qquad 0 \leqslant t \leqslant 1 .$$

a) Show that the process $(M_t, 0 \leqslant t < 1)$ defined by

$$M_t := \frac{S - X_t}{1-t}, \qquad 0 \leqslant t < 1 ,$$

is a càdlàg martingale with respect to the filtration $(\mathcal{F}_t, 0 \leqslant t < 1)$.

b) Is the martingale $(M_t, 0 \leqslant t < 1)$ uniformly integrable?

c) By introducing suitable truncations of the stopping time

$$T := \inf \{t \in [0,1] : 1 - S + X_t \leqslant t\} ,$$

or otherwise, show that

$$P(1 - S + X_t > t \qquad \text{for all} \quad t \in [0,1)) = 1 - S$$

[Hint: observe that $M_{T-} = 1$ whenever $T < 1$].

3 On some probability space $(\Omega, \mathcal{F}, P)$, let $(B_t\,, t \geqslant 0)$ be a standard real-valued Brownian motion. For $a > 0$ we let

$$\sigma_a = \inf\{t \geqslant 0 : |B_t| = a\}\,.$$

a) Show that for some constant $\rho \in (0,1)$, it holds that for every $n \geqslant 0$,

$$P(\sigma_1 > n) \leqslant \rho^n\,,$$

by noticing for instance that

$$\{\sigma_1 > n\} \subset \{|B_1| \leqslant 2\,,\ |B_2 - B_1| \leqslant 2\,,\ \ldots\,,\ |B_n - B_{n-1}| \leqslant 2\}\,.$$

Deduce that $E\left[(\sigma_1)^p\right] < \infty$ for every $p > 1$.

b) Show that there exists a constant $C > 0$ such that $E\left[\sigma_a\right] = Ca^2$ for every $a > 0$.

c) We define stopping times $(\sigma_a^n\,, n \geqslant 0)$ by

$$\sigma_a^0 = 0\,, \quad \sigma_a^1 = \sigma_a\,, \quad \sigma_a^{n+1} = \inf\{t \geqslant \sigma_a^n : |B_t - B_{\sigma_a^n}| = a\}\,.$$

Show that the variables $\sigma_a^{n+1} - \sigma_a^n$ are identically distributed. By computing the variance of $\sigma_{2^{-n}}^{2^{2n}}$ or otherwise, show that

$$\lim_{n\to\infty} \sigma_{2^{-n}}^{2^{2n}} = C \qquad \text{a.s.}$$

d) Show that the laws of the random variables $B_{\sigma^n_{1/\sqrt{n}}}\,, n \geqslant 1$ converge weakly as $n \to \infty$ to a limiting law to be made explicit. Deduce the exact value of C by comparing with part c).

4 On some probability space $(\Omega, \mathcal{F}, P)$, let $(B_t, t \geqslant 0)$ be a standard real-valued Brownian motion. For $t \geqslant 0$, we let $S_t = \sup_{0 \leqslant s \leqslant t} B_s$.

a) Show that $x^{-1} P(S_1 \leqslant x) \to c$ as $x \searrow 0$, for some constant $c > 0$.

b) We consider a function $f : (0, \infty) \to (0, \infty)$ which is increasing, continuous, and satisfies

$$\int_{(0,1]} f(t) \frac{dt}{t} < \infty .$$

Show that

$$\sum_{n \geqslant 0} P(S_{2^{-n-1}} < 2^{-n/2} f(2^{-n})) < \infty .$$

c) Deduce that, almost surely,

$$\liminf_{t \downarrow 0} \frac{S_t}{\sqrt{t}\, f(t)} \geqslant 1 ,$$

and hence show that this lim inf is in fact equal to ∞ a.s.

5 On some probability space $(\Omega, \mathcal{F}, P)$, let $(B_t, t \geqslant 0)$ be a standard Brownian motion taking its values in $\mathbb{R}^2$. We let λ be the Lebesgue measure on $\mathbb{R}^2$. For $y \in \mathbb{R}^2$, we let $T_y = \inf\{t \geqslant 0 : B_t = y\}$, and we aim to show, using only elementary properties of Brownian motion, that $\lambda(\{B_t : 0 \leqslant t \leqslant 1\}) = 0$ a.s. For simplicity, we will admit in this problem the fact that $E[\lambda(\{B_t : 0 \leqslant t \leqslant 1\})] < \infty$.

a) Let $A_1 = \{B_t : 0 \leqslant t \leqslant 1/2\}$ and $A_2 = \{B_t : 1/2 \leqslant t \leqslant 1\}$. Show that the random variables $\lambda(A_1)$ and $\lambda(A_2)$ have the same distribution, which is equal to that of

$$\frac{1}{2}\,\lambda\left(\{B_t : 0 \leqslant t \leqslant 1\}\right).$$

b) Deduce that

$$E\left[\lambda(\{B_t : 0 \leqslant t \leqslant 1/2\} \cap \{B_t : 1/2 \leqslant t \leqslant 1\})\right] = \int_{\mathbb{R}^2} \lambda(dy)\, P(y \in A_1 \cap A_2) = 0\,.$$

c) Show that the processes $(B_{1/2-t} - B_{1/2}\,,\ 0 \leqslant t \leqslant 1/2)$ and $(B_{t+1/2} - B_{1/2}\,,\ 0 \leqslant t \leqslant 1/2)$ are two independent standard Brownian motions defined on the time-interval $[0, 1/2]$. Deduce from b) that one has

$$\int_{\mathbb{R}^2} \lambda(dy)\, P(T_y \leqslant 1/2)^2 = 0\,,$$

and conclude that $E[\lambda(\{B_t : 0 \leqslant t \leqslant 1\})] = 0$.

6 On some probability space $(\Omega, \mathcal{F}, P)$, let M be a random point measure (countable sum of Dirac masses) on $\mathbb{R}_+ = [0, \infty)$, which a.s. assigns a finite mass to bounded sets and is simple, meaning that almost surely

$$M(\{t\}) \in \{0, 1\} \qquad \text{for every} \quad t \geqslant 0\,.$$

We assume that for every finite union A of bounded subintervals of $\mathbb{R}_+$, we have

$$P(M(A) \,=\, 0) \,=\, e^{-\lambda(A)}\,,$$

where λ is the Lebesgue measure on $\mathbb{R}_+$.

a) Let $I_1, \ldots, I_n$ be pairwise disjoint bounded subintervals of $\mathbb{R}_+$. Show that the events $\{M(I_i) = 0\}, 1 \leqslant i \leqslant n$ are independent.

b) For $n, k \geqslant 0$, we let $D_k^n = [k2^{-n}, (k+1)2^{-n})$. Fix J a bounded subinterval of $\mathbb{R}_+$, and let

$$M_n(J) \,=\, \sum_{k \geqslant 0 \,:\, D_k^n \subset J} \mathbb{1}_{\{M(D_k^n) \neq 0\}}\,.$$

(i) Show that $M_n(J)$ follows a Binomial distribution with parameters $(N_n\,,\, 1 - e^{-2^{-n}})$, where $N_n = \text{Card}\,\{k \geqslant 0 : D_k^n \subset J\}$.

(ii) Show that $M_n(J) \nearrow M(J)$ almost-surely as $n \nearrow \infty$, and deduce that $M(J)$ follows a Poisson distribution with mean $\lambda(J)$.

c) Let $J_1, \ldots, J_r$ be disjoint subintervals of $\mathbb{R}_+$. Show that $M_n(J_1), \ldots, M_n(J_r)$ are independent, and conclude that $M(J_1), \ldots, M(J_r)$ are independent. Deduce that, for every Borel function $f : \mathbb{R}_+ \to \mathbb{R}_+$, one has

$$E\left[\exp\left(-M(f)\right)\right] \,=\, \exp\left(-\int_{\mathbb{R}_+} \lambda(dx)\,(1 - e^{-f(x)})\right)\,,$$

and that M is a Poisson random measure on $\mathbb{R}_+$ with intensity λ.

END OF PAPER