UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2007 9.00 to 11.00

PAPER 32

ROUGH PATH THEORY AND APPLICATIONS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Define $\left(G^N\left(\mathbb{R}^d\right), \otimes, ^{-1}, e\right)$, the free step-$N$ nilpotent group over $\mathbb{R}^d$, and give the definition of the Carnot–Carathéodory d distance on $G^N\left(\mathbb{R}^d\right)$. What is a weak geometric p-rough path?

(ii) How can the step-2 nilpotent group over $\mathbb{R}^2$ be identified with the 3-dimensional Heisenberg group $\mathbb{H}$?

(iii) Since $\mathbb{H} \cong \mathbb{R}^3$ we can equip $\mathbb{H}$ with the *Euclidean* distance inherited from $\mathbb{R}^3$. Is a Lipschitz path in $\mathbb{H}$ relative to this Euclidean distance automatically a Lipschitz path relative to the Carnot–Carathéodory distance on $\mathbb{H}$?

2 Let x be a Lipschitz continuous $\mathbb{R}^d$-valued path. Define $S_N(x)_{s,t}$, the *step-N signature of the path segment* $x|_{[s,t]}$, as an element in a suitable tensor algebra over $\mathbb{R}^d$. State and prove an algebraic relation between the step-N signature of the path segment $x|_{[s,t]}$ and the path segment $x|_{[t,u]}$ respectively. Show that the signature is invariant under reparametrisation of the path. More precisely, given $\psi : [0,1] \to [0,1]$ strictly increasing and continuously differentiable, show that

$$S_N(x)_{0,1} = S_N(x \circ \psi)_{0,1}.$$

3 Nested piecewise linear approximations to d-dimensional Brownian motion and their canonical area converge to Brownian motion and Lévy area in a rough path sense. Give a precise statement of this and sketch a proof with particular focus on martingale arguments.

4 Write an essay on the rough path proof of the Stroock–Varadhan support theorem. In particular, explain how the universal limit theorem is used.

END OF PAPER