UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 7 June 2007 9.00 to 12.00

PAPER 30

ADDITIVE NUMBER THEORY

*Attempt **THREE** questions, exactly **ONE** of these being from Section A.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a function. Define the *Fourier transform* $\hat{f}(\xi)$. Suppose that

$$f = 1_{[0,1]} * \ldots * 1_{[0,1]},$$

the m-fold convolution of the (characteristic function of) the interval $[0,1]$ with itself. Show that

$$|\hat{f}(\xi)| \ll \min(1, |\xi|^{-m}).$$

Define the *Schwartz class* $\mathcal{S}(\mathbb{R})$, and show that the Fourier transform preserves this class. State and prove the Poisson summation formula. (Results on the uniqueness of Fourier series may be used without proof if stated correctly.)

Assuming (if you need it) a formula for the Fourier transform of a gaussian function, state and prove the functional equation of the Riemann ζ-function.

2 State the partial fraction expansion of the Riemann ζ-function.

Prove that there are no non-trivial zeros $\rho = \beta + i\gamma$ of ζ inside the region

$$\beta \geqslant 1 - c/\log(|\gamma| + 2)$$

for some absolute constant c.

State the explicit formula. Discuss how it can be used to prove the prime number theorem in the form

$$\sum_{n \leqslant X} \Lambda(n) = X + O(Xe^{-c\sqrt{\log X}}).$$

(Results on the existence of smooth bump functions with specified properties, on the decay of Γ'/Γ and on the decay of the Mellin transform in vertical strips may be stated without detailed proof.)

SECTION B

3 Prove that the sum of the reciprocals of the twin primes converges.

4 Let $k \geqslant 1$ be an integer, let p be a prime and suppose that $(a, p) = 1$. Define the Gauss sum in the usual way as $G_{a,p} := \mathbb{E}_{x \in \mathbb{Z}/p\mathbb{Z}} e(ax^k/p)$.

Establish the estimate

$$|G_{a,p}| \leqslant k/\sqrt{p}.$$

Using this estimate, show that if $p > p_0(k)$ is sufficiently large then every $x \in \mathbb{Z}/p\mathbb{Z}$ is a sum of three kth powers in $\mathbb{Z}/p\mathbb{Z}$.

5 Let $X = \{1^3, 2^3, \ldots, n^3\}$, where n is the largest integer such that $n^3 \leqslant N$.

Suppose that $\mathfrak{m}$ is a subset of $[0, 1)$ with the property that

$$\int_{\mathfrak{m}} |\hat{1}_X(\theta)|^9 \, d\theta \geqslant N^2/2007.$$

Prove that $\mathfrak{m}$ contains some number θ which is close to a rational in the sense that

$$\|q\theta\| \ll_\epsilon N^{\epsilon - 1}$$

for some $q \ll_\epsilon N^\epsilon$.

(If you use an "equidistribution lemma" you may state it without proof.)

6 Let $f : \{1, \ldots, N\} \to \mathbb{C}$ be a function with $|f(n)| \leqslant 1$ for all n. What does it mean for the Type I and Type II sums for f to be δ-small? Sketch, giving at least *some* of the more interesting technical details, a proof that if the Type I and Type II sums for f are $(\log^{-20} N)$-small then

$$\sum_{n \leqslant N} \Lambda(n) f(n) = o(N)\,.$$

Let $s(n)$ be the sum of the binary digits of n and set $f(n) = (-1)^{s(n)}$. Prove that the Type I sums for f are small.

END OF PAPER