UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2007 9.00 to 12.00

PAPER 3

MODULAR REPRESENTATIONS OF FINITE GROUPS

Attempt ***THREE*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

In this paper G is a finite group. In the usual notation we are given a p-modular system $(K, \mathfrak{O}, k)$ where p is a prime dividing $|G|$. Throughout $R \in \{\mathfrak{O}, k\}$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define a *block* of RG both in terms of ideal direct summands and idempotents. Explain how the two definitions are equivalent. Explain what it means for an indecomposable RG-module to *lie in* a block. Describe how the blocks over k and over $\mathfrak{O}$ are related.

Defining clearly any terms that you use, define the *defect group* and the *defect* of a block B. If B is a block with defect group D prove Green's result that every indecomposable RG-module in the block is projective relative to D. Finally, prove that if B is a block of kG and $\hat{B}$ is the corresponding block of $\mathfrak{O}G$ then B and $\hat{B}$ have the same defect groups.

2 State and prove Brauer's First Main Theorem (if you use any results in the proof you should state them clearly). If $DC_G(D) \leq H \leq G$ (but with no other restriction on H) and b is a p-block of kH, use the First Main Theorem to define the Brauer correspondent b^G of b. In this case prove that b^G may be characterised as the unique block B of kG such that $B\downarrow_{H\times H}$ has b as a direct summand viewed as a $k(H \times H)$-module.

3 For a fixed p-subgroup D of G, state and prove the Green Correspondence between blocks of RG and blocks of certain subgroups of G related to D. If you use any lemma you should prove it.

Take $R = k$. Use the Mackey Decomposition and Maschke's Theorem to give a direct proof of the Green Correspondence in the situation where a Sylow p-subgroup D of G is a T.I. set, namely, D satisfies $D \cap {}^gD = 1$ or D for $g \in G$. Namely, prove that if M is an indecomposable non-projective $kN_G(D)$-module then $M\uparrow^G$ has a unique non-projective indecomposable summand, i.e.

$$M\uparrow^G \cong M_0 \oplus M_1\,,$$

where M_1 is projective and M_0 is non-projective. [Standard facts about projective modules may be assumed.]

4 State Nagao's version of Brauer's Second Main Theorem.

Use it to deduce that if B is a block with defect group D then there exists an indecomposable kG-module lying in B with vertex D and a trivial source.

Recall that a block B is said to be of *finite representation type* if there are only finitely many isomorphism classes of indecomposable modules in B. Deduce that if B is a block with defect group D, then B has finite representation type if and only if D is cyclic.

Classify the indecomposable modules for a cyclic group of order p in characteristic p. Find the unique composition series for each indecomposable module and identify the principal indecomposable module.

5 Define the *principal block* of kG. State and prove Brauer's Third Main Theorem.

Denote the maximal normal p-subgroup of G by $O_p(G)$, and the maximal normal p'-subgroup (i.e. of order coprime to p) by $O_{p'}(G)$. Let $H = G/O_{p'}(G)$. We say that G is *p-constrained* if $C_H(O_p(H)) \leqslant O_p(H)$. Let $x \in G$ be a p-element such that $O_{p'}(C_G(x)) = 1$ and $C_G(x)$ is p-constrained. Let B be a block of kG with defect group D. Use the Third Main Theorem to prove that x is G-congugate to an element of D if and only if B is the principal block of G.

6 Let B be a p-block of kG with cyclic defect group D of order p^n ($n \geqslant 1$). Define the *inertial index* e of B. Assume that k is algebraically closed. Stating clearly any results you use, show that there are e simple modules in B and $p^n e$ indecomposable modules in B.

Let k be an algebraically closed field of characteristic 2. Suppose that B is a block of kG whose defect group is a Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Use the Extended First Main Theorem to show that $e = 1$ or 3. If $e = 1$ and D is normal in G prove that B is a complete matrix algebra $\mathrm{Mat}_n(kD)$ of some degree n. You should state clearly any results that you use. [HINT: it may be useful to consider separately the cases when D is central in G and when D is normal but not central in G.]

END OF PAPER