UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2007 1.30 to 3.30

PAPER 28

LOCAL FIELDS

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

$\mathbb{Q}_p$ denotes the field of p-adic numbers, $\mathbb{F}_p$ the field with p elements, and ζ_n a primitive nth root of unity.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Suppose K is a field, $v : K \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$ a normalised discrete valuation, and let $R = \{x \in K \mid v(x) \geq 0\}$.

(a) Prove that R is a ring, specifically a local integral domain, every ideal of R is principal, and K is the field of fractions of R.

(b) Denote the maximal ideal of R by m, pick a uniformiser π, and suppose A is a complete set of representatives for R/m. Prove that every non-zero element of K can be written uniquely as a series of the form

$$\sum_{n=n_0}^{\infty} a_n \pi^n, \qquad a_n \in A,\ a_{n_0} \notin m,$$

convergent in the topology of K induced by the valuation. Conversely, does every series of this form converge?

2 (a) State and prove a version of Hensel's lemma.

(b) Determine $\mathrm{Gal}(\mathbb{Q}_p(\zeta_8)/\mathbb{Q}_p)$ for every prime p.

3 (a) Suppose $(K, |\cdot|)$ is a complete non-Archimedean field. Prove that for every finite extension L/K, there is at most one extension of $|\cdot|$ to an absolute value on L. (You do not have to prove that such an extension exists.)

(b) Write down an absolute value $|\cdot|$ on $K = \mathbb{F}_p((t))$ whose ring of integers is $\mathbb{F}_p[[t]]$. Now let $e, f \geq 1$ be integers. Construct an extension L/K of residue degree f and ramification degree e, and determine explicitly the unique extension of $|\cdot|$ to it.

4 Let p be an odd prime.

(a) Show that the polynomial $\frac{(1+T)^p-1}{T} \in \mathbb{Q}_p[T]$ is Eisenstein, and deduce that $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p$ is totally ramified of degree $p-1$.

(b) Let $K = \mathbb{Q}_p(\zeta_p, \sqrt[p]{p})$. Prove that $K/\mathbb{Q}_p$ is Galois, totally ramified of degree $p(p-1)$, write down a uniformiser of K, and determine the size of every ramification subgroup G_i of $\mathrm{Gal}(K/\mathbb{Q}_p)$.

END OF PAPER