UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2007 9.00 to 12.00

PAPER 27

ELLIPTIC CURVES

*Attempt **ALL** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let E_d be the elliptic curve

$$X^3 + Y^3 + d\,Z^3 \;=\; 0$$

with identity element $(X : Y : Z) \;=\; (1 : -1 : 0)$.

(i) Describe the group law on E_d in terms of the chord and tangent process.

(ii) Find all the points of inflection on E_d defined over $\bar{\mathbb{Q}}$. What does this tell you about the torsion subgroup of $E_d(\mathbb{Q})$?

(iii) Show that E_d has Weierstrass equation $y^2 \;=\; x^3 - 432\,d^2$.

(iv) Show that there are infinitely many pairs of rational numbers x and y with $x^3 + y^3 \;=\; 7$.

2 (i) Let $f(t) \in R\,[[T]]$ be a power series with $f(0) = 0$ and $f'(0)$ a unit in the ring R. Show that there is a power series $g(t) \in R\,[[T]]$ with $f\,(g(T)) \;=\; g\,(f(T)) \;=\; T$.

(ii) Outline how the result in (i) is used in the proof of the weak Mordell-Weil theorem.

3 Either

(a) Write an essay on isogenies. You should include a proof that the degree map on $\mathrm{Hom}\,(E_1, E_2)$ is a positive definite quadratic form,

or

(b) Write an essay on heights and their application to the proof of the Mordell-Weil theorem.

[For either essay, you may find the following formulae useful :

Let $P_i = (x_i, y_i)$ for $i = 1, 2, 3, 4$ be non-zero points on the elliptic curve $y^2 \;=\; x^3 + ax + b$ with $P_3 \;=\; P_1 + P_2$ and $P_4 \;=\; P_1 - P_2$. Then

$$x_3 + x_4 \;=\; \frac{2\,(x_1x_2 + a)(x_1 + x_2) + 4b}{(x_1 - x_2)^2}\,,$$

$$x_3\,x_4 \;=\; \frac{x_1^2\,x_2^2 - 2a\,x_1\,x_2 - 4b\,(x_1 + x_2) + a^2}{(x_1 - x_2)^2}\,.\;\Big]$$

4 Describe a procedure, that often works in practice, to compute the rank of an elliptic curve over $\mathbb{Q}$ with a rational 2-torsion point. Show that if p is an odd prime then the elliptic curve $y^2 = x^3 - p^2x$ has rank at most 2. Give an example of a prime p for which the rank is exactly 1.

END OF PAPER