UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2007 9.00 to 12.00

PAPER 25

CATEGORY THEORY

*Attempt **ONE** question from **Section I** and **TWO** questions from **Section II**.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1 **Either** State and prove the General Adjoint Functor Theorem, and use it to prove that, for any functor $F : \mathcal{C} \to \mathcal{D}$ between small categories, the functor $F^* : [\mathcal{D}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$ induced by composition with F has a left adjoint;

Or State and prove the Special Adjoint Functor Theorem, and use it to prove that, for any functor $F : \mathcal{C} \to \mathcal{D}$ between small categories, the functor $F^* : [\mathcal{D}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$ induced by composition with F has a right adjoint.

[The Yoneda Lemma, and standard results on limits and colimits in functor categories, may be assumed.]

2 'Category Theory is the one area of mathematics where definitions matter more than theorems.' Write an essay arguing the case **either** for **or** against this statement, illustrating your argument with definitions and/or theorems drawn from the course.

SECTION II

3 A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *final* if, for each object B of $\mathcal{D}$, the arrow category $(B \downarrow F)$ is (nonempty and) connected. F is said to be a *discrete fibration* if, given $A \in \mathrm{ob}\,\mathcal{C}$ and $f : B \to FA$ in $\mathcal{D}$, there is a unique $\widetilde{f} : \widetilde{B} \to A$ in $\mathcal{C}$ with $F\widetilde{f} = f$.

(i) Show that if we are given a commutative square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{H} & \mathcal{C} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle G} \\ \mathcal{B} & \xrightarrow{K} & \mathcal{D} \end{array}$$

where F is final and G is a discrete fibration, then there is a unique functor $L : \mathcal{B} \to \mathcal{C}$ with $LF = H$ and $GL = K$.

(ii) Show that any functor $F : \mathcal{C} \to \mathcal{D}$ can be factored as a final functor followed by a discrete fibration.

[Hint: construct a category whose objects are all connected components of the categories $(B \downarrow F)$, $B \in \mathrm{ob}\,\mathcal{D}$.]

(iii) Deduce from (i) that the factorization in (ii) is unique up to canonical isomorphism.

4 Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$. Show that $T\eta = \eta_T$ if and only if μ is a natural isomorphism. [A monad with these properties is said to be *idempotent*.]

Now suppose $\mathcal{C}$ has equalizers; for each object A of $\mathcal{C}$, let $\beta_A : RA \rightarrowtail TA$ denote the equalizer of η_{TA} and $T\eta_A$. Show that R may be made into a functor $\mathcal{C} \to \mathcal{C}$ in such a way that β becomes a natural transformation $R \to T$. Show also that the morphisms $\alpha_A : A \to RA$ and $\theta_A : RRA \to RA$, which are the factorizations through β_A of η_A and the composite

$$RRA \xrightarrow{R\beta_A} RTA \xrightarrow{\beta_{TA}} TTA \xrightarrow{\mu_A} TA$$

respectively, give R the structure of a monad $\mathbb{R} = (R, \alpha, \theta)$.

[You are not required to justify the existence of these factorizations.]

Show further that α_{TA} is an isomorphism for all A. [Standard results on split equalizer diagrams may be assumed.] By considering the diagram

$$TA \xrightarrow{T\alpha_A} TRA \xrightarrow{T\beta_A} TTA \underset{TT\eta_A}{\overset{T\eta_{TA}}{\rightrightarrows}} TTTA$$

show that $T\alpha_A$ is an isomorphism iff $T\beta_A$ is monic. If these latter conditions hold, show further that the square

$$\begin{array}{ccc} RRA & \xrightarrow{R\beta_A} & RTA \\ \downarrow{\scriptstyle \beta_{RA}} & & \downarrow{\scriptstyle \beta_{TA}} \\ TRA & \xrightarrow{T\beta_A} & TTA \end{array}$$

is a pullback, and deduce that the monad $\mathbb{R}$ is idempotent.

5 Define the notions of *additive category* and *abelian category*. Prove that finite products and coproducts coincide in any additive category.

A poset P is said to be *directed* if it is nonempty and, for any two elements x, y of P, there exists $z \in P$ with $x \leqslant z$ and $y \leqslant z$. By a *directed diagram* in a category, we mean one whose shape is a directed poset. We say a cocomplete abelian category $\mathcal{A}$ is *finitary* if, given a directed diagram D in $\mathcal{A}$ and a cone under D whose legs are all monic, the induced morphism from the colimit of D to the summit of the cone is also monic. Show that the category **AbGp** is finitary.

[Hint: first show that the forgetful functor **AbGp** $\to$ **Set** creates colimits of directed diagrams.]

Now suppose $\mathcal{A}$ is a complete and cocomplete abelian category. If $\mathcal{A}$ is finitary, show that the canonical morphism

$$\sum_{i \in I} A_i \longrightarrow \prod_{i \in I} A_i$$

represented by the 'infinite identity matrix' is monic for any family of objects $(A_i \mid i \in I)$. [Hint: represent the infinite coproduct as a directed colimit of finite coproducts.] Now suppose that $\mathcal{A}$ is both finitary and cofinitary; by considering the morphism

$$A \xrightarrow{\ \Delta\ } \prod_{i \in N} A \longrightarrow \sum_{i \in N} A \xrightarrow{\ \nabla\ } A$$

where Δ and ∇ are the diagonal and codiagonal maps, and the middle factor is the inverse of the morphism considered earlier, deduce that $\mathcal{A}$ is degenerate (i.e., all its objects are zero objects).

6 Define the notion of *monoidal category*, and state and prove the coherence theorem for monoidal categories.

END OF PAPER