UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2007 9.00 to 11.00

PAPER 24

THREE–MANIFOLDS

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Given two non-zero coprime integers p and q, define the lens space $\mathcal{L}(p,q)$. Write down its fundamental group, giving a brief justification for your answer.

(b) Explain how to associate a 3–manifold M_A to a matrix

$$A = \begin{pmatrix} -q & s \\ p & r \end{pmatrix}$$

in $SL(2,\mathbb{Z})$. Prove carefully that $M_A \cong \mathcal{L}(p,q)$ whenever p and q are non-zero coprime integers.

(c) Prove that the lens spaces $\mathcal{L}(7,2)$ and $\mathcal{L}(7,3)$ are homeomorphic.

2 (a) Define what it means for a closed connected orientable 3–manifold M to be (i) irreducible; (ii) prime; (iii) geometrizable; and (iv) of Heegaard genus g.

(b) Prove that $S^1 \times S^2$ is prime but not irreducible, and that it is the only closed connected orientable 3–manifold with this property.

(c) Let H_1 and H_2 be two handlebodies each of genus two. Suppose that α and β are two simple closed curves on ∂H_1 each of which bounds an embedded disc in H_1, such that cutting H_1 along these discs produces a 3-ball. Similarly, let γ and δ be two simple closed curves on ∂H_2 each of which bounds an embedded disc in H_2, such that cutting H_2 along these discs produces a 3-ball. Suppose that ϕ is a homeomorphism from ∂H_1 to ∂H_2 such that $\phi(\alpha) = \gamma$ and $\phi(\beta)$ is disjoint from γ and intersects δ once. Consider an orientable manifold M of Heegaard genus two which is obtained by gluing H_1 to H_2 via this homeomorphism ϕ. Prove that such a manifold M is geometrizable.

You should quote carefully any results that you use.

3 (a) Define carefully what is meant by (i) two embeddings of S^1 in S^3 being ambient isotopic; (ii) a link diagram of a link of a finite disjoint union of copies of S^1 in S^3.

(b) Define the Kauffman bracket $\langle D \rangle$ of a link diagram D. Giving D an orientation, define the Jones polynomial of D and prove carefully that it is an invariant of the oriented link represented by D.

(c) Let K denote the four-crossing figure-eight knot embedded in S^3. Prove that its Jones polynomial in $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ is symmetric under the map $t \mapsto t^{-1}$.

(d) Calculate the fundamental group of the figure-eight knot complement $S^3 - K$, simplifying this to a one-relator presentation.

4 (a) Let S be a closed orientable surface of genus $g \geqslant 1$ and let h be an orientation-preserving homeomorphism of S to itself. Define what it means for h to be a Dehn twist.

(b) State precisely the Dehn–Lickorish theorem concerning homeomorphisms of a compact orientable surface S. Use this theorem to prove the Fundamental Theorem of Dehn Surgery.

(c) Let L be the link in S^3 consisting of two linked but unknotted circles and suppose that N_L is a small regular neighbourhood of L consisting of two disjoint solid tori U_1 and U_2. For each $i \in \{1, 2\}$ let α_i be a non-trivial simple closed curve on ∂U_i such that α_i bounds an embedded disc in $S^3 - U_i$. Explain why each α_i is unique up to isotopy.

(d) Let M be the manifold obtained from $S^3 - N_L$ by attaching solid tori V_1 and V_2 to ∂N_L such that the meridian of V_i is identified with α_i, for $i = 1$ and $i = 2$. What is M? Justify your answer.

5 (a) Let Γ be a finitely generated group. Define what it means for Γ to be a Kleinian group.

(b) For a finitely generated Kleinian group Γ, define the limit set $\Lambda(\Gamma)$ of Γ and prove that this definition is independent of any arbitrary choices you may have made.

(c) We say that a Kleinian group is non-elementary if $\Lambda(\Gamma)$ contains at least three points. Let Γ be a non-elementary Kleinian group and write S^2_∞ for the boundary sphere at infinity of $\mathbb{H}^3$. Prove that every non-empty closed subset of S^2_∞ invariant under the action of Γ contains $\Lambda(\Gamma)$.

(d) Deduce that any infinite proper normal subgroup Γ' of Γ satisfies $\Lambda(\Gamma') = \Lambda(\Gamma)$.

(e) Give an example of a finitely generated non-elementary Kleinian group Γ for which $\mathbb{H}^3/\Gamma$ is **not** a hyperbolic 3–manifold, briefly justifying your answer.

END OF PAPER