UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2007 9.00 to 12.00

PAPER 22

COMPLEX MANIFOLDS

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define an almost complex structure J on a real manifold M. Explain briefly how J induces the type decomposition of the complexified tangent bundle $TM \otimes \mathbb{C}$ and define the differential (p, q)-forms on M.

Show that the following conditions (a) and (b) are equivalent:

(a) for every two vector fields X, Y of type $(1, 0)$ on M, the vector field $[X, Y]$ also has type $(1, 0)$;

(b) if α is a $(1, 0)$-form on M then $d\alpha$ is the sum of a $(2, 0)$-form and a $(1, 1)$-form.

An almost complex structure J is said to be *integrable* if the condition (a) or (b) holds.

[*Basic properties of real differential forms and real vector fields may be used without proof if accurately stated.*]

Now suppose that M is a $2n$-dimensional real manifold endowed with a complex n-form Ω, such that $\Omega \wedge \overline{\Omega}$ is nowhere-zero, where $\overline{\Omega}$ denotes the complex conjugate of Ω . Suppose also that each point $x \in M$ has a neighbourhood U_x such that $\Omega|_{U_x} = \theta_1 \wedge \ldots \wedge \theta_n$, for some complex 1-forms θ_k on U_x. Show that there is a uniquely determined almost complex structure J_Ω on M, so that any θ_k as above is a $(1, 0)$-form. By considering the expressions $d\theta_k \wedge \Omega$, or otherwise, show that if $d\Omega = 0$ then J_Ω is integrable.

2 Define the differential operators ∂ and $\bar{\partial}$ for the differential (p, q)-forms on a complex manifold X. Explain what is meant by holomorphic p-forms.

Give definitions of a holomorphic vector bundle, transition functions, and holomorphic sections. Show that the bundle of $(1, 0)$-forms on a complex manifold is a holomorphic bundle and identify, giving justification, the space of all the holomorphic sections of this bundle.

Define the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{C}P^n$ and show that it is a holomorphic bundle. Show that $\mathcal{O}(-1)$ has no non-zero holomorphic sections over $\mathbb{C}P^n$.

[*You may use without proof auxiliary results on vector bundles over smooth manifolds provided that you state these correctly.*]

3 Define the terms *irreducible subvariety of codimension* k in a compact complex manifold X and *divisors* on X.

Explain what is meant by a local defining function of a divisor D on X and by the associated holomorphic line bundle $[D]$, showing that $[D]$ is well-defined. You should state clearly the auxiliary properties of local rings of holomorphic functions that you require. Prove the isomorphism of line bundles $[D_1+D_2] \cong [D_1]\otimes[D_2]$, for any two divisors D_1, D_2 on X.

Let Y be a non-singular hypersurface in a complex manifold X. Prove that the restriction of the line bundle $[-Y]$ to Y is isomorphic to the conormal bundle of Y.

[*You may assume that transition functions determine a vector bundle up to an isomorphism.*]

4 Define the blow-up $\tilde{X}$ of a complex manifold X at a point $p \in X$ and the exceptional divisor E on $\tilde{X}$. Construct a family of coordinate charts on $\tilde{X}$ near points of E, in the case when X is a polydisc in $\mathbb{C}^n$.

State and prove the relation between the canonical bundles of X and $\tilde{X}$, assuming that K_X has a meromorphic section which is not identically zero.

[*Standard results on the line bundles associated to divisors may be assumed if accurately stated.*]

Given a point $x \in \mathbb{C}P^2$, show that the assignment to each $y \neq x$ of the complex projective line passing through x and y induces a holomorphic map $f : \mathbb{C}P^2 \setminus \{x\} \to \mathbb{C}P^1$. Show further that there is a holomorphic map $\tilde{f} : S \to \mathbb{C}P^1$ such that $f \circ \sigma(z) = \tilde{f}(z)$ whenever $\sigma(z) \neq x$, where $\sigma : S \to \mathbb{C}P^2$ is the blow-up of $\mathbb{C}P^2$ at x.

5 Let X be a compact Hermitian manifold. Define the associated $(1,1)$-form ω of the Hermitian metric on X and write down the volume form of the induced Riemannian metric. Define the Hodge star operator for the differential (p,q)-forms on X and compute $*\omega$. Define the Laplacians Δ_d and $\Delta_{\bar{\partial}}$ on X and state the Hodge theorem for $\Delta_{\bar{\partial}}$.

Show that the (point-wise) adjoint Λ of the Lefschetz operator $L(\alpha) = \omega \wedge \alpha$ on the differential forms on X satisfies $\Lambda\beta = (-1)^{p+q} * L * \beta$, for each (p,q)-form β.

[*You may assume that* $**\alpha = (-1)^{p+q}\alpha$, *for each* (p,q)*-form* α.]

Now suppose that the Hermitian metric on X is Kähler. Assuming the identity $[\Lambda, \partial] = i\bar{\partial}^*$, deduce the relation $\Delta_d = 2\Delta_{\bar{\partial}}$. Show that the Betti numbers $b^{2k-1}(X)$ are even and $b^{2k}(X) > 0$, for $k = 1, \ldots, \dim X$. Give an example, with a brief justification, of a compact complex manifold which does not admit any Kähler metrics.

[*You may assume that the space of* d*-harmonic* r*-forms on a compact oriented Riemannian manifold is isomorphic to the* r*-th de Rham cohomology.*]

END OF PAPER