UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 12 June 2007 9.00 to 12.00

PAPER 2

QUANTUM GROUPS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Define $SL_q(2)$ and give its coalgebra and algebra structure explicitly.

(b) Define the coaction of $SL_q(2)$ on $k_q[x, y]$ and compute explicitly $\triangle x^2 y$.

(c) Explain what is meant by an R-point of $k_q[x, y]$. What are the $\mathbf{C}$ points of $k_q[x, y]$? If R is the algebra $M_n(\mathbf{C})$ of n by n matrices, and α is an R-point of $k_q[x, y]$, show that α determines a decomposition

$$\mathbf{C}^n = V_x \oplus V_y \oplus U$$

where V_x is the subspace on which $\alpha(x)$ has non-zero eigenvalues, V_y is the subspace on which $\alpha(y)$ has non-zero eigenvalues, and U is the subspace on which both $\alpha(x)$ and $\alpha(y)$ act nilpotently. [*Hint: In particular, you must show $\alpha(y)$ acts nilpotently on V_x.*] You may assume q is not a root of unity.

2 (a) Let V be a 3 dimensional simple U_q module where q is not a root of unity. Show that there is a basis of V with respect to which K, E, F are represented by the following matrices:

$$E = \epsilon \begin{pmatrix} 0 & [2] & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & 0 & 0 \\ [2] & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$K = \epsilon \begin{pmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{pmatrix}$$

where $\epsilon = \pm 1$.

(b) Decompose $V_{1,1} \otimes V_{1,2}$ into its simple U_q modules, indicating which are highest weight vectors, and giving bases explicitly. You may use a different basis from that in part (a) if you prefer.

3 (a) Explain what is meant by a cobraided bialgebra.

(b) Describe the Faddeev-Reshitikin-Takhtadjian construction. If $V =< e_1, e_2 >$, and $C : V \otimes V \to V \otimes V$ is given by

$$C = \begin{pmatrix} q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & q^{-\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} & q^{-\frac{1}{2}}(q - q^{-1}) & 0 \\ 0 & 0 & 0 & q^{\frac{1}{2}} \end{pmatrix}$$

define an isomorphism from $M_q(2)$ to the FRT algebra, and check $ba = qab$.

(c) Give the cobraiding r explicitly.

(d) Calculate explicitly $r(a^2 \otimes b)$.

4 (a) Define the action of U_q on $k_q[x, y]$, and show explicitly that the Serre relations hold.

(b) Show that the space of homogeneous polynomials of degree n, $k_q^n[x, y]$ is a submodule of $k_q[x, y]$.

(c) Now suppose that $q^3 = 1$. Show that U_q has no simple submodule of dimension greater than 3.

(d) What are the simple submodules of $k_q^3[x, y]$? Can $k_q^3[x, y]$ be expressed as a direct sum of simple submodules? Explain.

5 (a) If A, B are algebras, define what is meant by a measuring coalgebra for the pair A, B. Define (by stating its universal property) what is meant by the universal measuring coalgebra $P(A, B)$.

(b) Let C_q be the comodule given by

$$C_q = < K, K^{-1}, I, E, F >$$

with K, K^{-1}, and I all group-like, and comultiplication of E, F given by

$$\triangle F = F \otimes I + K^{-1} \otimes F\,,$$

$$\triangle E = E \otimes K + I \otimes E\,.$$

If

$$p : C_q \longrightarrow End(k_q[x, y])$$

where

$$p(K)(x) = qx, \quad p(K)(y) = q^{-1}y$$
$$p(K^{-1})(x) = q^{-1}x, \quad p(K^{-1})(y) = qy$$
$$p(E)(y) = x,\; p(E)(x) = 0\,,$$
$$p(F)(x) = y, \quad p(F)(y) = 0\,,$$

and p is a measuring map, show that

$$p(E)x^r x^s = [s]x^{r+1}y^{s-1}\,.$$

(c) Outline the proof that there is a bialgebra homomorphism

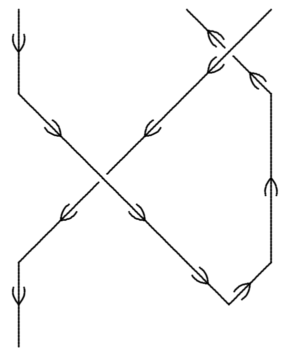
$$\rho : U_q \longrightarrow P(k_q[x, y],\, k_q[x, y])\,.$$

6 (a) Define the tangle category.

(b) Draw representatives of the classes:

- $(\uparrow \cup \downarrow \overleftarrow{\cap} X_-)$
- $(\cup) \circ (\downarrow \cap \uparrow) \circ (X_+ \uparrow\uparrow)$

(c) Write the tangle represented by



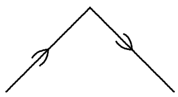
in terms of elementary tangles

$$\downarrow, \uparrow, \cap, \overleftarrow{\cap}, \cup, \overleftarrow{\cup}, X_+, X_- \,.$$

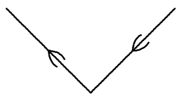
(d) Let $V =< u, v >$ and $V^* =< x, y >$. Let c be the matrix

$$c = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 0 & q^{-2} & 0 \\ 0 & q^{-2} & q^{-2}(q - q^{-1}) & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}$$

with respect to the basis $u \otimes u,\ u \otimes v,\ v \otimes u,\ v \otimes v$ of $V \otimes V$. Suppose that under a functor $\mathcal{F}$ elementary tangles are represented by maps as follows



$$= \quad 1 \mapsto (x \otimes u + y \otimes v),$$



$$= \quad \begin{matrix} x \otimes u \mapsto A \\ y \otimes v \mapsto B \\ x \otimes v \mapsto 0 \\ y \otimes u \mapsto 0, \end{matrix}$$

$$X_+ = c.$$

Compute A and B, and verify that the quantum dimension of V is [2]. [*Hint: consider an identity involving these tangles.*)

END OF PAPER