UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 7 June 2007 1.30 to 4.30

PAPER 19

COBORDISM

*Attempt **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\Omega^*_U(\cdot)$ be complex cobordism, i.e. the cobordism theory corresponding to stable complex structures in vector bundles. Explain what is meant by a complex structure on a continuous map $f : X \to Y$ of C^∞-smooth manifolds, and for a fixed complex structure on f define (without proofs) the Gysin map $f_!$ in $\Omega^*_U(\cdot)$. Prove that if L is a submanifold of a manifold M and the normal bundle ν of the embedding $i : L \subset M$ has a stable complex structure, then

$$i^* i_!(1) \in \Omega^{2n}_U(L)$$

is the top Chern class of ν in complex cobordism ($\dim_{\mathbb{C}} \nu = n$). You may assume that all manifolds in this question are compact and without a boundary.

2 Let η be a vector bundle, $\dim_{\mathbb{R}} \eta = n$, over a smooth base space X and with a framed structure, i.e. with a continuous choice of an (ordered) orthonormal frame in each fiber of η. By considering the appropriate (non-ordered) Stiefelization of η, for each $k = 1, 2, \ldots, n$ construct an $\binom{n}{k}$-sheeted cover $p_k : X_k \to X$, and define exotic characteristic classes of η by

$$l_k(\eta) = (p_k)_!\,(1) \in \Omega^0_{fr}(X), \quad k = 1, \ldots, n$$

Deduce the Whitney sum formula for l_k:

$$l_k(\eta \oplus \zeta) = \sum_{i+j=k} l_i(\eta) l_j(\zeta),$$

for two framed bundles η and ζ.

3 Define the d_1-metric on the space of C^∞-maps of a compact manifold $M \subset \mathbb{R}^k$ into the Euclidean space $\mathbb{R}^N$. Let $f_1, f_2, \ldots \in C^\infty(M, \mathbb{R}^N)$ be a sequence of maps which converges with respect to the d_1-metric to an embedding $i : M \subset \mathbb{R}^N$. Prove that there is an N_0 such that for any $n > N_0$ the map $f_n : M \to \mathbb{R}^N$ is an immersion. Assuming that the second derivatives of all the f_n, $n = 1, 2, \ldots$ are bounded by a constant C, prove that there is an N' such that for any $n > N'$ the map $f_n : M \to \mathbb{R}^N$ is an embedding.

4 State the axiom of exactness in a generalized cohomology theory $h^*(\cdot)$. Explain why it follows from exactness that for a one-point space x_0 the group $h^n(x_0, x_0)$ is trivial for any n. Define the wedge product of two CW-pairs (X, A) and (Y, B). Assuming that $h^*(\cdot)$ is multiplicative, explain in which group the product of two elements $x \in h^k(X, A)$ and $y \in h^m(Y, B)$ lies. Let $A_1, A_2, \cdots, A_l$ be subcomplexes of a pointed CW-complex (X, x_0) such that

$$X = \bigcup_{j=1}^{l} A_j,$$

and suppose each A_j is contractible (homotopy equivalent to a point). Prove that for any $w_1, \ldots, w_l \in h^*(X, x_0)$ the product $w_1 \cdot w_2 \cdot \cdots \cdot w_l$ is zero.

5 Let $\mathbb{H}P^2$ be the quaternionic projective plane. Compute the complex cobordism ring $\Omega_U^*(\mathbb{H}P^2, \emptyset)$ as a ring over $\Omega_U^*(\{pt\}, \emptyset)$. Explain carefully all steps of your proof.

END OF PAPER