UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2007 1.30 to 4.30

PAPER 16

ALGEBRAIC TOPOLOGY

Attempt ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Give a basis for the integral homology of $\mathbb{C}P^2 \times \mathbb{C}P^2$. Compute the class of the diagonal submanifold $\mathbb{C}P^2 \subset \mathbb{C}P^2 \times \mathbb{C}P^2$ in $H_4(\mathbb{C}P^2 \times \mathbb{C}P^2, \mathbb{Z})$.

2

Describe a cell decomposition of $\mathbb{R}P^2 \times \mathbb{R}P^2$. Compute the cellular chain complex of $\mathbb{R}P^2 \times \mathbb{R}P^2$ (with integer coefficients). Use this chain complex to compute the integral homology of $\mathbb{R}P^2 \times \mathbb{R}P^2$.

3

Let F be a topological space, and let $\pi : E \to B$ be a continuous map of topological spaces. Say that an open subset $U \subset B$ is *evenly covered* if there is a homeomorphism $Q : \pi^{-1}(U) \to U \times F$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "Q"] \arrow[dr] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

commutes. Suppose that every point of B is contained in some evenly covered open subset. [You may use that every open subset of an evenly covered open set is evenly covered.]

Also, assume that there are classes $c_1, \ldots, c_v \in H^*(E, \mathbb{Q})$, $c_j \in H^{k_j}(E, \mathbb{Q})$, whose restrictions to $\pi^{-1}(p) \cong F$ form a basis for $H^*(F, \mathbb{Q})$ as a vector space, for each point $p \in B$. Write i for the inclusion map from $F = \pi^{-1}(p)$ into E. Show that, for B compact, the map

$$H^*(B, \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(F, \mathbb{Q}) \to H^*(E, \mathbb{Q}),$$

$\sum_{k,l} b_k \otimes i^*(c_l) \longmapsto \sum_{k,l} \pi^*(b_k) \cup c_l$, is an isomorphism.

[*Hint: First prove the theorem when B is replaced by a single evenly covered open set V and E is replaced by $\pi^{-1}(U)$.*]

4

Define a 3-manifold X by starting with $S^2 \times [0,1]$ and identifying $S^2 \times \{0\}$ with $S^2 \times \{1\}$ by the antipodal map. Is X orientable?

Compute the integral cohomology groups of X. Determine the cup products on the integral cohomology. From there (or otherwise), compute the cohomology ring of X with $\mathbb{Z}/2$ coefficients.

5

(i) Show that the symmetric group S_3 can act freely on the closed orientable surface of genus 7. [*Hint: what might the quotient space be?*]

(ii) Define the cap product $\cap : H_*(X,R) \times H^*(X,R) \to H_*(X,R)$, for a space X and a commutative ring R. (Do *not* check that it is well-defined.) For a continuous map $f : X \to Y$, prove that

$$f_*\left(x \cap f^*(\alpha)\right) = f_*(x) \cap \alpha$$

for all $x \in H_*(X,R)$ and $\alpha \in H^*(Y,R)$.

END OF PAPER