UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2007 9.00 to 12.00

PAPER 15

DIFFERENTIAL GEOMETRY

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let M be a smooth manifold.

(a) Define the tangent bundle TM of M and show that it can be endowed in a natural way with a smooth structure making it a vector bundle over M.

(b) Define the cotangent bundle T^*M of M and show that it can be endowed in a natural way with a smooth structure making it a vector bundle over M.

(c) Are TM and T^*M isomorphic as vector bundles? Justify your answer.

2 Let (M, g) be an n-dimensional Riemannian manifold and ∇ its Levi-Civita connection.

(a) Let X be a smooth vector field on M. Define the *divergence* of X as the function $\operatorname{div} X : M \to \mathbf{R}$ given by $\operatorname{div} X(p) = \operatorname{trace}(v \mapsto (\nabla_v X)(p))$.

Fix a point $p \in M$. Suppose that there exist a neighbourhood U of p and n smooth vector fields $E_1, \ldots, E_n$ defined on U such that $\{E_1, \ldots, E_n\}$ is an orthonormal basis at every point of U and $(\nabla_{E_i} E_j)(p) = 0$ for all i and j. Show that $\operatorname{div} X(p) = \sum_{i=1}^n E_i(f_i)(p)$ where $X = \sum_{i=1}^n f_i E_i$.

(b) Suppose M is orientable and let ω_g be its Riemannian volume form. For $i = 1, \ldots, n$, let ω_i be the 1-forms on U defined by $\omega_i(E_j) = \delta_{ij}$, where $\{E_1, \ldots, E_n\}$ is a frame as in (a) which is compatible with the orientation of M. Consider the $(n-1)$-forms $\theta_i := \omega_1 \wedge \cdots \wedge \widehat{\omega}_i \wedge \cdots \wedge \omega_n$, where $\widehat{\omega}_i$ indicates that the form ω_i is omitted from the product. Show that

$$\nu = \sum_i (-1)^{i+1} f_i \, \theta_i$$

where ν is the $(n-1)$-form defined by $\nu(Y_1, \ldots, Y_{n-1}) = \omega_g(X, Y_1, \ldots, Y_{n-1})$.

(c) Show that $d\nu = (\operatorname{div} X)\, \omega_g$.

[*You may assume that for any 1-form* α, $d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y])$.]

(d) Suppose M is compact and orientable and let f be any smooth function. Show that

$$\int_M X(f)\, \omega_g = -\int_M f \operatorname{div} X \, \omega_g.$$

3 Let A be a connection on a vector bundle E.

(a) Using local coordinates on the base manifold and a local trivialization of E, give an explicit local formula for the covariant derivative d_A induced by A and acting on the sections of E. Explain how to extend d_A, using an appropriate version of the Leibnitz rule, to the differential forms with values in E and to the differential forms with values in the endomorphism bundle $\operatorname{End} E$. For both cases, include explicit formulas for d_A in local trivializations.

(b) Define the curvature F of a connection A, showing that F is a well-defined 2-form with values in $\operatorname{End} E$. Prove that if σ be an E-valued r-form, then $d_A^2(\sigma) = F \wedge \sigma$.

(c) Prove the Bianchi identity $d_A F = 0$. By using the Bianchi identity or otherwise, show that if E is a vector bundle of rank 1, then F is a closed form and its de Rham cohomology class is independent of the choice of connection A.

[*Preliminary results on connections may be used without proof provided these are clearly stated.*]

4 Let M be a smooth n-dimensional manifold.

(a) Let $\widehat{M}$ be the set of pairs (p, o_p), where $p \in M$ and o_p is one of the two orientations of T_pM. Let $\pi : \widehat{M} \to M$ be $\pi(p, o_p) = p$. Given an open oriented set $U \subset M$ with orientation form $\omega \in \Omega^n(U)$ we let $\widehat{U} \subset \widehat{M}$ be the set of pairs (p, o_p), where $p \in U$ and o_p is the orientation of T_pM determined by ω_p. Show that $\widehat{M}$ has a topology such that $\widehat{U}$ is open and π maps $\widehat{U}$ homeomorphically onto U for every oriented open set $U \subset M$.

(b) Show that $\widehat{M}$ has a smooth structure such that π maps $\widehat{U}$ diffeomorphically onto U for every oriented open set $U \subset M$. Show that $\widehat{M}$ has a canonical orientation.

(c) Let M be a connected smooth manifold. Show that $\widehat{M}$ has at most two connected components and that M is orientable if and only if $\widehat{M}$ is not connected.

5 (a) State the Hodge decomposition theorem.

(b) Let M be a compact Riemannian manifold with $\operatorname{Ric} \geqslant 0$. Show that any harmonic 1-form is parallel. [*You may assume that if ω is a harmonic 1-form and X is the vector field dual to ω then,*

$$-\Delta\left(|\omega|^2/2\right) = |\nabla\omega|^2 + \operatorname{Ric}(X, X),$$

where Δ is the Laplace-Beltrami operator.]

(c) Show that a compact connected Riemannian manifold of dimension n with $\operatorname{Ric} \geqslant 0$ has first Betti number less than or equal to n. Give an example of a compact manifold of dimension $\geqslant 3$ which does not admit a metric of non-negative Ricci curvature.

END OF PAPER