UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2007 1.30 to 3.30

## PAPER 14

## RAMSEY THEORY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

(i) Using Ramsey's theorem, show that, whenever $\mathbb{N}$ is finitely coloured, there exist $x_1 < x_2 < x_3 < \ldots$ such that the set $\{x_i + x_j : i \neq j\}$ is monochromatic.

[No form of Hindman's theorem may be assumed.]

(ii) Show that whenever $\mathbb{N}$ is finitely coloured, there exist $x_1 < x_2 < x_3 < \ldots$ such that the set $\{x_i + 2x_j : i < j\}$ is monochromatic.

(iii) Show that it is *not* true that, whenever $\mathbb{N}$ is finitely coloured, there exist $x_1 < x_2 < x_3 < \ldots$ such that the set $\{x_i + 2x_j : i \neq j\}$ is monochromatic.

(iv) Deduce from (iii) that there is no ultrafilter on $\mathbb{N}$, each member of which contains a set of the form $\{x_i + 2x_j : i \neq j\}$ (where $x_1 < x_2 < x_3 < \ldots$).

**2**

State and prove van der Waerden's theorem. Deduce that, if $a_1, \ldots, a_n$ are non-zero rationals, then the matrix $(a_1, \ldots, a_n)$ is partition regular if and only if some (non-empty) subset of the $a_i$ has sum zero.

[No form of Rado's theorem may be assumed without proof.]

**3**

What is an *ultrafilter* on $\mathbb{N}$? Prove that there exists a non-principal ultrafilter on $\mathbb{N}$. Define the topological space $\beta\mathbb{N}$, and prove that it is compact and Hausdorff.

State Hindman's theorem, and show how to deduce it from the existence of an idempotent for $+$ on $\beta\mathbb{N}$. (You are not required to prove that an idempotent exists. You may assume simple properties of ultrafilters, their quantifiers, and the operation $+$ on $\beta\mathbb{N}$).

Deduce from Hindman's theorem the following statement: whenever $\mathbb{N}$ is finitely coloured, there exist $x_1 < x_2 < x_3 < \ldots$ such that $FS(x_1, x_2, x_3, \ldots)$ is monochromatic and also $x_i$ divides $x_{i+1}$ for all $i$.

**4**

What does it mean to say that a subset of $\mathbb{N}^{(\omega)}$ is *Ramsey*? Give an example of a set that is not Ramsey. Prove that every $\tau$-open set is Ramsey.

Find, with justification, examples of each of the following:

(i) a set that is $*$-open but not $\tau$-open,

(ii) a set that is $\tau$-nowhere-dense but not $*$-nowhere-dense,

(iii) a set that is $*$-nowhere-dense but not $\tau$-nowhere-dense.

## END OF PAPER