UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2007 1.30 to 3.30

PAPER 14

RAMSEY THEORY

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Using Ramsey's theorem, show that, whenever $\mathbb{N}$ is finitely coloured, there exist $x_1 < x_2 < x_3 < \ldots$ such that the set $\{x_i + x_j : i \neq j\}$ is monochromatic.

[No form of Hindman's theorem may be assumed.]

(ii) Show that whenever $\mathbb{N}$ is finitely coloured, there exist $x_1 < x_2 < x_3 < \ldots$ such that the set $\{x_i + 2x_j : i < j\}$ is monochromatic.

(iii) Show that it is *not* true that, whenever $\mathbb{N}$ is finitely coloured, there exist $x_1 < x_2 < x_3 < \ldots$ such that the set $\{x_i + 2x_j : i \neq j\}$ is monochromatic.

(iv) Deduce from (iii) that there is no ultrafilter on $\mathbb{N}$, each member of which contains a set of the form $\{x_i + 2x_j : i \neq j\}$ (where $x_1 < x_2 < x_3 < \ldots$).

2

State and prove van der Waerden's theorem. Deduce that, if $a_1, \ldots, a_n$ are non-zero rationals, then the matrix $(a_1, \ldots, a_n)$ is partition regular if and only if some (non-empty) subset of the a_i has sum zero.

[No form of Rado's theorem may be assumed without proof.]

3

What is an *ultrafilter* on $\mathbb{N}$? Prove that there exists a non-principal ultrafilter on $\mathbb{N}$. Define the topological space $\beta\mathbb{N}$, and prove that it is compact and Hausdorff.

State Hindman's theorem, and show how to deduce it from the existence of an idempotent for $+$ on $\beta\mathbb{N}$. (You are not required to prove that an idempotent exists. You may assume simple properties of ultrafilters, their quantifiers, and the operation $+$ on $\beta\mathbb{N}$).

Deduce from Hindman's theorem the following statement: whenever $\mathbb{N}$ is finitely coloured, there exist $x_1 < x_2 < x_3 < \ldots$ such that $FS(x_1, x_2, x_3, \ldots)$ is monochromatic and also x_i divides x_{i+1} for all i.

4

What does it mean to say that a subset of $\mathbb{N}^{(\omega)}$ is *Ramsey*? Give an example of a set that is not Ramsey. Prove that every τ-open set is Ramsey.

Find, with justification, examples of each of the following:

(i) a set that is $*$-open but not τ-open,

(ii) a set that is τ-nowhere-dense but not $*$-nowhere-dense,

(iii) a set that is $*$-nowhere-dense but not τ-nowhere-dense.

END OF PAPER