UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2007 9.00 to 11.00

PAPER 12

COMBINATORICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove the Kruskal-Katona theorem.

Deduce that if $\mathcal{A} \subset [n]^{(r)}$ is an intersecting system and $r \leqslant n/2$ then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.

Give an example of a system $\mathcal{A} \subset [n]^{(2)}$ that is not isomorphic to the initial colex segment $\mathcal{I}$ of $[n]^{(2)}$ with $|\mathcal{A}| = |\mathcal{I}|$, but is such that $|\partial\mathcal{A}| = |\partial\mathcal{I}|$. [That is, $\mathcal{A}$ is not equivalent to $\mathcal{I}$ under some permutation of $[n]$.] Give a corresponding example of a system $\mathcal{A} \subset [n]^{(3)}$.

2 Let $\mathcal{A} \subset [n]^{(r)}$. Show that if m is a prime power, and $|A \cap B| \not\equiv r \pmod{m}$ for all distinct A, $B \in \mathcal{A}$, then $|\mathcal{A}| \leq \binom{n}{m-1}$.

Either derive two significant applications of this result,

or show how it might fail if m is not a prime power.

3 State and prove the Ahlswede-Khachatrian theorem about $M(n, r, t)$, the maximum size of a t-intersecting family $\mathcal{A} \subset [n]^{(r)}$.

[You may use a lemma about the existence of a generating family on a small ground-set, provided you state it clearly and indicate the main ideas of its proof.]

4 Let $\mathcal{A} \subset \mathcal{P}[n]$. For $i \in [n]$ let $\mathcal{A}(i) = \{A \setminus \{i\} : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : A \setminus \{i\} \in \mathcal{A}\}$. Moreover, given $Y \subset [n]$, let $\mathcal{A}_{|Y} = \{Y \cap A : A \in \mathcal{A}\}$.

Prove that $\left|A(i)_{|Y}\right| \leq \left|A_{|Y}\right|$ for all $Y \subset [n]$.

State and prove the Sauer-Shelah lemma on families that shatter k-sets. Does there exist a family $\mathcal{A} \neq [n]^{(\leq 2)}$, with $|\mathcal{A}| = 1 + n + \binom{n}{2}$, that shatters no set $Y \in [n]^{(3)}$?

Prove that, if $|\mathcal{A}| \leq \lceil 3n/2 \rceil$ then $\left|A_{|Y}\right| \geq |\mathcal{A}| - 1$ for some $Y \in [n]^{(n-1)}$, but that this need not be true if $|\mathcal{A}| \geq \lceil 3n/2 \rceil + 1$.

END OF PAPER