

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2007 9.00 to 11.00

PAPER 12

COMBINATORICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

*Cover sheet
Treasury Tag
Script paper*

SPECIAL REQUIREMENTS

None

<p>You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.</p>

1 State and prove the Kruskal-Katona theorem.

Deduce that if $\mathcal{A} \subset [n]^{(r)}$ is an intersecting system and $r \leq n/2$ then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.

Give an example of a system $\mathcal{A} \subset [n]^{(2)}$ that is not isomorphic to the initial colex segment \mathcal{I} of $[n]^{(2)}$ with $|\mathcal{A}| = |\mathcal{I}|$, but is such that $|\partial\mathcal{A}| = |\partial\mathcal{I}|$. [That is, \mathcal{A} is not equivalent to \mathcal{I} under some permutation of $[n]$.] Give a corresponding example of a system $\mathcal{A} \subset [n]^{(3)}$.

2 Let $\mathcal{A} \subset [n]^{(r)}$. Show that if m is a prime power, and $|A \cap B| \not\equiv r \pmod{m}$ for all distinct $A, B \in \mathcal{A}$, then $|\mathcal{A}| \leq \binom{n}{m-1}$.

Either derive two significant applications of this result,

or show how it might fail if m is not a prime power.

3 State and prove the Ahlswede-Khachatrian theorem about $M(n, r, t)$, the maximum size of a t -intersecting family $\mathcal{A} \subset [n]^{(r)}$.

[You may use a lemma about the existence of a generating family on a small ground-set, provided you state it clearly and indicate the main ideas of its proof.]

4 Let $\mathcal{A} \subset \mathcal{P}[n]$. For $i \in [n]$ let $\mathcal{A}(i) = \{A \setminus \{i\} : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : A \setminus \{i\} \in \mathcal{A}\}$. Moreover, given $Y \subset [n]$, let $\mathcal{A}|_Y = \{Y \cap A : A \in \mathcal{A}\}$.

Prove that $|\mathcal{A}(i)|_Y| \leq |\mathcal{A}|_Y|$ for all $Y \subset [n]$.

State and prove the Sauer-Shelah lemma on families that shatter k -sets. Does there exist a family $\mathcal{A} \neq [n]^{(\leq 2)}$, with $|\mathcal{A}| = 1 + n + \binom{n}{2}$, that shatters no set $Y \in [n]^{(3)}$?

Prove that, if $|\mathcal{A}| \leq \lceil 3n/2 \rceil$ then $|\mathcal{A}|_Y| \geq |\mathcal{A}| - 1$ for some $Y \in [n]^{(n-1)}$, but that this need not be true if $|\mathcal{A}| \geq \lceil 3n/2 \rceil + 1$.

END OF PAPER