UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 7 June 2007 1.30 to 4.30

PAPER 11

RIEMANN SURFACES AND DISCRETE GROUPS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let D be a proper subdomain of the complex plane. For each point $z_o \in D$ set

$$\delta(z_o) = \inf\{|z_o - w| : w \in \mathbb{C} \setminus D\} \ .$$

Show that there is a point $w_o \in \mathbb{C} \setminus D$ with $\delta(z_o) = |z_o - w_o|$.

The function

$$f(z) = \frac{z - z_o}{z - w_o}$$

is holomorphic. The principal branch of its logarithm is given by the convergent power series

$$\log f(z) = \sum_{k=1}^{\infty} -\frac{1}{k}\left(\frac{z_o - w_o}{z - w_o}\right)^k$$

on the set $\{z : |z - w_o| > \delta(z_o)\}$. Does it converge uniformly on this set? Does it converge locally uniformly?

For any $\epsilon_o > 0$, show that there is a natural number K for which the function

$$E_o(z) = \left(\frac{z - z_o}{z - w_o}\right) \exp\left[\sum_{k=1}^{K} \frac{1}{k}\left(\frac{z_o - w_o}{z - w_o}\right)^k\right]$$

is holomorphic on D, has a simple zero at z_o and satisfies

$$|\log E_o(z)| < \epsilon_o \qquad \text{for} \qquad |z - w_o| > 2\delta(z_o) \ .$$

Deduce that, for any sequence (z_n) of distinct points in D that satisfies $\delta(z_n) \to 0$ as $n \to \infty$, there is a holomorphic function $f : D \to \mathbb{C}$ that has simple zeros at each z_n and no other zeros.

Explain why this implies that every meromorphic function on D can be written as the quotient of two holomorphic functions on D.

2 Let $u : \overline{\mathbb{D}} \to \mathbb{R}$ be a continuous function on the closed unit disc $\overline{\mathbb{D}}$ which is harmonic on the open unit disc. Show that the value of u at any point $z \in \mathbb{D}$ is given by a Poisson integral of the boundary values $u(w)$ for $w \in \partial\mathbb{D}$.

A continuous function $u : \Omega \to \mathbb{R}$ on a domain $\Omega \subset \mathbb{C}$ has the *mean value property* if, for each $z \in \Omega$ there exists $r(z) > 0$ with $\{w : |w - z| < r(z)\} \subset \Omega$ and

$$u(z) = \int_0^{2\pi} u(z + re^{i\theta})\, \frac{d\theta}{2\pi}$$

for $0 < r < r(z)$. Prove that, if such a function has a local maximum at $z \in \Omega$, then it is constant on a neighbourhood of z. Prove that u has the mean value property if, and only if, u is harmonic.

3 Let (z_n) be a sequence of points in the unit disc $\mathbb{D}$ and ρ the hyperbolic metric on $\mathbb{D}$. Prove that the following conditions are equivalent.

(a) For a point $w \in \mathbb{D}$ the sum $\sum \exp -\rho(w, z_n)$ converges.

(b) There is a bounded holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with f having a zero of order k at any point $z \in \mathbb{D}$ if and only if z occurs exactly k times in the sequence (z_n).

Let G be a discrete subgroup of the group $\text{Möb}(\mathbb{D})$ of all Möbius transformations from the disc onto itself. Let B be a Blaschke product for the sequence of points in an orbit $G(0) = \{T(0) : T \in G\}$. Show that there is a group homomorphism

$$\chi : G \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

with

$$B(T(z)) = \chi(T)B(z) \qquad \text{for all } z \in \mathbb{D} \text{ and } T \in G\ .$$

4 Let G be the set of all Möbius transformations of the form

$$z \mapsto \frac{az + b}{\bar{b}z + \bar{a}}$$

where $|a|^2 - |b|^2 = 1$ and $a, b \in \mathbb{Z}[i] = \{m + ni : m, n \in \mathbb{Z}\}$. Explain briefly why G acts discontinuously on the unit disc.

Let A and B be the Möbius transformations

$$A : z \mapsto \frac{(1-i)z + i}{-iz + (1+i)} \qquad ; \qquad B : z \mapsto \frac{(1-i)z - i}{iz + (1+i)}$$

and let H be the group they generate. Show that this is a discrete group. Find the fixed points of A and B and show that both A and B are parabolic transformations.

Let ρ denote the hyperbolic metric on $\mathbb{D}$. Show that the set

$$\{z \in \mathbb{D} : \rho(z, 0) \leqslant \rho(z, A^k(0)) \text{ for } k \in \mathbb{Z}\}$$

is a subset of $\mathbb{D}$ bounded by two hyperbolic geodesics. Draw these geodesics on a diagram. What is the corresponding result for B?

Let $H(z_o)$ be the orbit of any point $z_o \in \mathbb{D}$. Show that there is an element T of H with $\rho(0, T(z_o))$ minimal and that the point $T(z_o)$ lies in a region of $\mathbb{D}$ bounded by four hyperbolic geodesics. Identify the quotient $\mathbb{D}/H$ up to homeomorphism.

5 Define a *Perron family* of continuous subharmonic functions on a Riemann surface R. Prove that the supremum of such a Perron family is either $+\infty$ on all of R or else a harmonic function on R. Give examples to show that both cases arise.

Let $u : \mathbb{D} \to \mathbb{R}$ be continuous and subharmonic on the unit disc $\mathbb{D}$. Show that the least harmonic majorant of u is given by

$$\lim_{r \to 1-} \int_0^{2\pi} u(re^{i\theta}) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \, \frac{d\theta}{2\pi} = \sup_{r<1} \int_0^{2\pi} u(re^{i\theta}) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \, \frac{d\theta}{2\pi}.$$

6 What does it mean to say that a simply-connected Riemann surface is *hyperbolic.* Give an example of such a surface and prove that it is indeed hyperbolic.

Write an essay describing the proof that a simply-connected, hyperbolic, Riemann surface is conformally equivalent to the unit disc.

END OF PAPER