UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2007 1.30 to 4.30

PAPER 10

ANALYSIS OF OPERATORS

Attempt ***THREE*** *questions.*

There are ***EIGHT*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let V be a finite–dimensional complex inner product space and let $W = \Lambda(V)$. For $v \in V$ define the exterior multiplication operator $e(v)$ on W. Prove that $e(a)e(b)^* + e(b)^*e(a) = (a,b)1$ for $a, b \in V$. Show that the operators $e(a)$, $e(b)^*$ $(a, b \in V)$ act irreducibly on W.

(b) Let V be a finite–dimensional complex inner product space and $A \subseteq \mathrm{End}(V)$ a unital *–algebra. Define the commutant A' of A and prove that $A'' = A$.

(c) Explain what it means for a strong operator continuous homomorphism $z \mapsto U_z$, $\mathbb{T} \to U(H)$ to have *positive energy*. If in addition $\mathcal{S} \subset B(H)$ satisfies $\mathcal{S}^* = \mathcal{S}$, $\mathcal{S} = U_z \mathcal{S} U_z^*$ for all $z \in \mathbb{T}$, and H is irreducible for $\mathcal{S} \cup \{U_z\}$, prove that any operator that commutes with $\mathcal{S}$ necessarily commutes with any operator U_z.

2 (a) Let A be the Banach algebra $H_s(S^1)$ for $s > 1/2$ and let A_+ and A_- be the closed subalgebras given by the vanishing of negative and positive Fourier coefficients. Prove that if $X \in GL_n(A)$ and $\|X - I\| < 1$, then there are unique $X_\pm \in GL_n(A_\pm)$ such that $X = X_- X_+$ and $X_+(0) = I$.

(b) Let $\mathcal{G}(H)$ be the group of invertible operators on the Hilbert space H of the form $I + T$ with T trace–class. Given a differentiable map $F : (a, b) \to \mathcal{G}(H)$, prove that $f(t) = \det F(t)$ is differentiable with

$$f^{-1}\dot{f} = \mathrm{Tr}(F^{-1}\dot{F}).$$

(c) If $A, B \in B(H)$ with $[A, B]$ trace–class, prove that $e^A e^B e^{-A} e^{-B}$ lies in $\mathcal{G}(H)$ with

$$\det(e^A e^B e^{-A} e^{-B}) = \exp \mathrm{Tr}\,(AB - BA).$$

(d) Prove that if $f \in C^\infty(S^1)$ with $f(z) = \sum a_n z^n$, then

$$\det T(e^f)T(e^{-f}) = \exp \sum_{n>0} n a_n a_{-n},$$

where $T(g)$ is the Toeplitz operator with symbol g.

3 (a) On $L^2(\mathbb{R})$ write down a family of unitary operators $W(x,y)$ $(x, y \in \mathbb{R})$ satisfying

$$W(x_1,y_1)W(x_2,y_2) = e^{i(x_1y_2-y_1x_2)}W(x_1+x_2,y_1+y_2).$$

Prove that an operator commuting with all the $W(x,y)$'s is necessarily a scalar operator.

(b) Prove that any strong operator continuous unitary map $(x,y) \mapsto W(x,y)$, satisfying these relations and acting irreducibly, is unitarily equivalent to the map given in (a).

(c) Show that if $a,b,c,d \in \mathbb{R}$ satisfy $ad-bc=1$, then there is a unitary V on $L^2(\mathbb{R})$, unique up to a phase, such that $VW(x,y)V^* = W(ax+by, cx+dy)$ for all $x,y \in \mathbb{R}$. When $a=d=0$, $b=1$ and $c=-1$, find a corresponding operator V.

4 (a) State and prove the Jacobi triple product formula.

(b) Prove that every strong operator continuous homomorphism of $\mathbb{T}$ into the projective unitary group $PU(H)$ lifts to a continuous homomorphism into $U(H)$.

(c) Prove that every strong operator continuous homomorphism of $\mathbb{R}$ into $PU(H)$ lifts to a continuous homomorphism into $U(H)$ (you may assume that any unitary U can be written e^{iA} for A self–adjoint in U'').

5 Write an essay on the index of Fredholm operators. You should include in your account a discussion of Toeplitz operators on the circle.

6 Write an essay on Sobolev spaces and eigenfunction expansions for second order elliptic operators on $\mathbb{T}^n$.

7 Write an essay on the Fourier transform on $\mathbb{R}^n$. Your essay should include a discussion of creation and annhihilation operators and a proof that the Fourier transform induces a bijection on $\mathcal{S}(\mathbb{R}^n)$. (You may assume the Stone–Weierstrass theorem if needed.)

8 Write an essay on positive energy representations of the loop group of $U(1)$ and the fermion–boson correspondence. Your account should contain a discussion of Segal's quantisation criterion and the 2–cocycle on the loop group.

END OF PAPER