UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2007 9.00 to 12.00

# PAPER 1

# SMOOTH REPRESENTATION THEORY OF $p$-ADIC GROUPS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

*In the following questions, $F$ always denotes a non-Archimedean local field with ring of integers $\mathfrak{o}$ and maximal ideal $\mathfrak{p} = \varpi\mathfrak{o}$. The valuation $|\cdot|$ on $F$ is normalized by $|\varpi| = q^{-1}$, where $q$ is the cardinality of the residue field of $F$. The notation $\mathrm{diag}(a_1, \ldots, a_n)$ denotes an $n$-by-$n$ square matrix $(a_{ij})$ all of whose non-diagonal entries $a_{ij}$ are zero, and $a_{ii} = a_i$, $i = 1, \ldots, n$.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** (a) Let $\iota : F \to \mathbb{C}$ be an embedding of fields and $n \geqslant 1$ a natural number. Is the induced homomorphism $\pi : GL_n(F) \to GL_n(\mathbb{C})$, $(a_{ij}) \mapsto (\iota(a_{ij}))$, a smooth representation? Explain your answer.

(b) Let $G$ be an $\ell$-group, $V = C_c^\infty(G, \mathbb{C})$ the space of locally constant functions with compact support on $G$. Show that the representation $\rho : G \to GL(V)$, $(\rho(g)f)(x) = f(xg)$, is smooth. Show furthermore that it is admissible if and only if $G$ is compact.

**2** For each natural number $i \geqslant 1$ let $\chi_i$ be a complex-valued character of $F^\times$ which is trivial on $1 + \mathfrak{p}^{i+1}$ but not trivial on $1 + \mathfrak{p}^i$. Let $(V_i, \chi_i)$ be the one-dimensional representation of $F^\times$ on $\mathbb{C}$ given by the character $\chi_i$.

(a) Let $(V, \pi)$ be the representation of $F^\times$ which is the direct sum of the representations $(V_i, \chi_i)$, i.e.

$$V = \bigoplus_{i \geqslant 1} V_i \, .$$

Is $V$ an admissible representation of $F^\times$? Explain your answer.

(b) Show that the representation $\pi^*$ on the *algebraic* dual space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is naturally isomorphic to

$$\prod_{i \geqslant 1} V_i^* \, ,$$

where $V_i^*$ is the one-dimensional representation given by the character $\chi_i^{-1}$.

(c) Show that the smooth dual $V^\vee$ of $V$, i.e. the subrepresentation of $V^*$ consisting of all smooth vectors, is

$$V^\vee = \bigoplus_{i \geqslant 1} V_i^* \, .$$

**3** Let $n \geqslant 2$ be a natural number, $B \subset GL_n(F)$ be the subgroup of upper-triangular matrices, $U \subset B$ the normal subgroup of upper-triangular matrices having 1's on the diagonal, and $T \subset B$ the subgroup of diagonal matrices. Let $(V, \rho)$ be an admissible representation of $B$.

(a) Let $\delta = \mathrm{diag}(\varpi^{-(n-1)}, \varpi^{-(n-2)}, \ldots, 1)$, and $K \subset B$ be a compact-open subgroup of the form $T_0 U_0$ with compact-open subgroups $T_0 \subset T$ and $U_0 \subset U$. For $i \geqslant 0$ put $K_i = \delta^i K \delta^{-i}$. Show that the map

$$V^K \to V^{K_i} \, , \; v \mapsto \rho(\delta^i)(v) \, ,$$

is an isomorphism. Show further that $K$ is contained in $K_i$ for $i \gg 0$. Using this and a dimension argument, deduce that $V^K = V^{K_i}$ for all sufficiently large $i$.

(b) Use (a) to prove that $U$ acts trivially on $V$.

**4** Let $G = GL_2(F)$, $B \subset G$ the subgroup of upper-triangular matrices, $U \subset B$ the normal subgroup of upper-triangular matrices having 1's on the diagonal, and $T \subset B$ the subgroup of diagonal matrices. For a character $\chi$ of $T$ denote by $V(\chi)$ the parabolically induced representation $\mathrm{Ind}_B^G(\chi)$. (As usual, $\chi$ is regarded as a character of $B$ via the canonical homomorphism $B \to T$).

(a) Prove that for two characters $\chi, \xi$ of $T$ the space

$$\mathrm{Hom}_G(V(\chi), V(\xi))$$

is one-dimensional if and only if $\xi = \chi$ or $\xi = \chi^w \delta^{-1}$, and zero otherwise.

Here, $\chi^w(\mathrm{diag}(t_1, t_2)) = \chi(\mathrm{diag}(t_2, t_1))$ and $\delta(\mathrm{diag}(t_1, t_2)) = |t_2/t_1|$.

You may use without proof that the Jacquet-module $V(\chi)_U$ of $V(\chi)$ sits in an exact sequence

$$0 \to \chi^w \delta^{-1} \to V(\chi)_U \to \chi \to 0\,.$$

(b) Denote by $\mathbf{1}$ the trivial character of $T$, as well as the trivial one-dimensional representation of $G$. It is known that the induced representation $V(\mathbf{1})$ sits in an exact sequence

$$0 \to \mathbf{1} \to V(\mathbf{1}) \to \mathrm{St} \to 0\,,$$

where the representation on the right hand side is the Steinberg representation. Use (a) to show that this sequence does *not* split, i.e. $V(\mathbf{1})$ is *not* isomorphic to

$$\mathbf{1} \oplus \mathrm{St}\,.$$

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