UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2007 9.00 to 12.00

PAPER 1

SMOOTH REPRESENTATION THEORY OF P-ADIC GROUPS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

In the following questions, F always denotes a non-Archimedean local field with ring of integers $\mathfrak{o}$ and maximal ideal $\mathfrak{p} = \varpi\mathfrak{o}$. The valuation $|\cdot|$ on F is normalized by $|\varpi| = q^{-1}$, where q is the cardinality of the residue field of F. The notation $\mathrm{diag}(a_1, \ldots, a_n)$ denotes an n-by-n square matrix (a_{ij}) all of whose non-diagonal entries a_{ij} are zero, and $a_{ii} = a_i$, $i = 1, \ldots, n$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let $\iota : F \to \mathbb{C}$ be an embedding of fields and $n \geqslant 1$ a natural number. Is the induced homomorphism $\pi : GL_n(F) \to GL_n(\mathbb{C})$, $(a_{ij}) \mapsto (\iota(a_{ij}))$, a smooth representation? Explain your answer.

(b) Let G be an ℓ-group, $V = C_c^\infty(G, \mathbb{C})$ the space of locally constant functions with compact support on G. Show that the representation $\rho : G \to GL(V)$, $(\rho(g)f)(x) = f(xg)$, is smooth. Show furthermore that it is admissible if and only if G is compact.

2 For each natural number $i \geqslant 1$ let χ_i be a complex-valued character of $F^\times$ which is trivial on $1 + \mathfrak{p}^{i+1}$ but not trivial on $1 + \mathfrak{p}^i$. Let (V_i, χ_i) be the one-dimensional representation of $F^\times$ on $\mathbb{C}$ given by the character χ_i.

(a) Let (V, π) be the representation of $F^\times$ which is the direct sum of the representations (V_i, χ_i), i.e.

$$V = \bigoplus_{i \geqslant 1} V_i \,.$$

Is V an admissible representation of $F^\times$? Explain your answer.

(b) Show that the representation π^* on the *algebraic* dual space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is naturally isomorphic to

$$\prod_{i \geqslant 1} V_i^* \,,$$

where V_i^* is the one-dimensional representation given by the character χ_i^{-1}.

(c) Show that the smooth dual $V^\vee$ of V, i.e. the subrepresentation of V^* consisting of all smooth vectors, is

$$V^\vee = \bigoplus_{i \geqslant 1} V_i^* \,.$$

3 Let $n \geqslant 2$ be a natural number, $B \subset GL_n(F)$ be the subgroup of upper-triangular matrices, $U \subset B$ the normal subgroup of upper-triangular matrices having 1's on the diagonal, and $T \subset B$ the subgroup of diagonal matrices. Let (V, ρ) be an admissible representation of B.

(a) Let $\delta = \mathrm{diag}(\varpi^{-(n-1)}, \varpi^{-(n-2)}, \ldots, 1)$, and $K \subset B$ be a compact-open subgroup of the form $T_0 U_0$ with compact-open subgroups $T_0 \subset T$ and $U_0 \subset U$. For $i \geqslant 0$ put $K_i = \delta^i K \delta^{-i}$. Show that the map

$$V^K \to V^{K_i} \,, \; v \mapsto \rho(\delta^i)(v) \,,$$

is an isomorphism. Show further that K is contained in K_i for $i \gg 0$. Using this and a dimension argument, deduce that $V^K = V^{K_i}$ for all sufficiently large i.

(b) Use (a) to prove that U acts trivially on V.

4 Let $G = GL_2(F)$, $B \subset G$ the subgroup of upper-triangular matrices, $U \subset B$ the normal subgroup of upper-triangular matrices having 1's on the diagonal, and $T \subset B$ the subgroup of diagonal matrices. For a character χ of T denote by $V(\chi)$ the parabolically induced representation $\mathrm{Ind}_B^G(\chi)$. (As usual, χ is regarded as a character of B via the canonical homomorphism $B \to T$).

(a) Prove that for two characters χ, ξ of T the space

$$\mathrm{Hom}_G(V(\chi), V(\xi))$$

is one-dimensional if and only if $\xi = \chi$ or $\xi = \chi^w \delta^{-1}$, and zero otherwise.

Here, $\chi^w(\mathrm{diag}(t_1, t_2)) = \chi(\mathrm{diag}(t_2, t_1))$ and $\delta(\mathrm{diag}(t_1, t_2)) = |t_2/t_1|$.

You may use without proof that the Jacquet-module $V(\chi)_U$ of $V(\chi)$ sits in an exact sequence

$$0 \to \chi^w \delta^{-1} \to V(\chi)_U \to \chi \to 0\,.$$

(b) Denote by $\mathbf{1}$ the trivial character of T, as well as the trivial one-dimensional representation of G. It is known that the induced representation $V(\mathbf{1})$ sits in an exact sequence

$$0 \to \mathbf{1} \to V(\mathbf{1}) \to \mathrm{St} \to 0\,,$$

where the representation on the right hand side is the Steinberg representation. Use (a) to show that this sequence does *not* split, i.e. $V(\mathbf{1})$ is *not* isomorphic to

$$\mathbf{1} \oplus \mathrm{St}\,.$$

END OF PAPER