UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 13 June, 2006 9 to 12

PAPER 88

MODULAR FORMS

*Attempt **at most FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

We use the following notations thoughout: $\Lambda \subset \mathbb{C}$ is a lattice, τ belongs to the upper half-plane.

For any $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$ *we write* $f\big|[\alpha]_k(\tau) = \det(\alpha)^{k-1}(c\tau + d)^{-k} f(\alpha(\tau))$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the Weierstrass $\wp$-function and show that it has Laurent expansion

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty}(2k-1)G_{2k}z^{2k-2}$$

where as usual

$$G_{2k} = \sum_{0\neq\omega\in\Lambda}\frac{1}{\omega^{2k}}.$$

(You may assume the convergence of the relevant series.) Deduce that $\wp(z)$ satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where $g_2 = 60G_4$ and $g_3 = 140G_6$.

By considering the Laurent expansion of $\wp''(z)$ or otherwise, show that for $k > 3$, G_{2k} may be expressed as a polynomial in G_4 and G_6 with positive, rational coefficients.

2 (i) Let M, N be positive integers with $M|N$. Show that if D is a divisor of N/M then for any $f \in S_k(\Gamma_0(M))$ the function $f(D\tau)$ belongs to $S_k(\Gamma_0(N))$.

(ii) Let p be prime. Show that $\Delta(p\tau) \in S_{12}(\Gamma_0(p))$, and that the orders of $\Delta(\tau)$, $\Delta(p\tau)$ at the cusps of $\Gamma_0(p)$ are given by the table

	$\Delta(\tau)$	$\Delta(p\tau)$
cusp ∞	1	p
cusp 0	p	1

(iii) Assuming that $S_2(\Gamma_0(11))$ has dimension 1, show that an element of this space is $(\Delta(\tau)\Delta(11\tau))^{1/12}$.

3 Let the operator W_N be defined on modular forms of weight k by

$$W_N f = i^k N^{1-k/2} f \Big| \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}_k$$

Show that W_N maps $S_k(\Gamma_1(N))$ to itself, and that W_N^2 is the identity map.

Suppose that $f = \sum a_n q^n \in S_k(\Gamma_1(N))$ satisfies $W_N f = \epsilon f$ where $\epsilon \in \{\pm 1\}$. Show that the completed L-function

$$\Lambda_N(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

has an analytic continuation to the complex plane, and satisfies the functional equation $\Lambda_N(f, k-s) = \epsilon \Lambda_N(f,s)$.

4 Use the Poisson summation formula to show that the function $\vartheta(\tau) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau)$ satisfies the transformation rule $\vartheta(-1/\tau) = \sqrt{-i\tau}\,\vartheta(\tau)$.

Use the Mellin transform of $(\vartheta(it)-1)/2$ to obtain the functional equation for the Riemann ζ-function.

5 For congruence subgroups Γ_1, Γ_2 of $SL_2(\mathbb{Z})$, and $\alpha \in GL_2^+(\mathbb{Q})$ define the double coset operator $[\Gamma_1 \alpha \Gamma_2]$ on $M_k(\Gamma_1)$, and show that its image is contained in $M_k(\Gamma_2)$.

Define the operators $<d>$ and T_p on $M_k(\Gamma_1(N))$, for d and p prime to N. Show that they commute, and that if $a_n(f)$ are the Fourier coefficients of $f \in M_k(\Gamma_1(N))$, then for every p not dividing N,

$$a_n(T_p f) = a_{np}(f) + p^{k-1} a_{n/p}(<p> f).$$

END OF PAPER