UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 12 June, 2006 9 to 11

PAPER 86

FOURIER TRANSFORMS, THEIR GENERALISATIONS AND THE IMAGING OF THE BRAIN

*Attempt **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

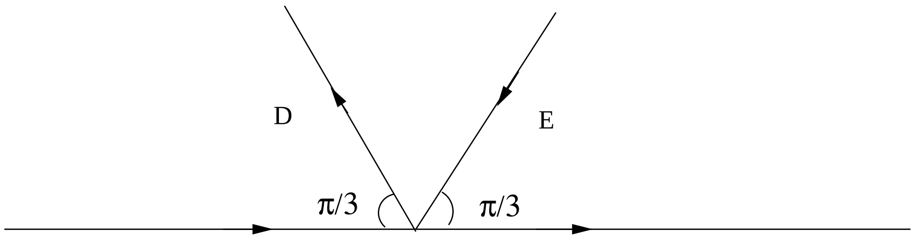
You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) By assuming the validity of the Fourier transform, prove the validity of the following transform pair:

$$\hat{q}(k) = \int_0^\infty e^{-ikx} q(x)dx,$$

$$q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{q}(k)dk + \frac{c_1}{2\pi}\int_{\partial E} e^{ikx}\hat{q}(\alpha^2 k)dk + \frac{c_2}{2\pi}\int_{\partial D} e^{ikx}\hat{q}(\alpha k)dk,$$

where $\alpha = \exp[2i\pi/3]$, c_1, c_2 are constants and the contours ∂D and ∂E are the boundaries of the domains D and E with the orientation shown below, $D : k \in \mathbf{C}$, $\arg k \in [\frac{2\pi}{3}, \pi]$, $E : k \in \mathbf{C}$, $\arg k \in [0, \frac{\pi}{3}]$.



The domains D and E.

(b) Let $q(x,t)$ solve any of the following three initial-boundary value problems:

$$q_t - q_{xxx} = 0, \quad 0 < x < \infty, \quad t > 0,$$

$$q(x,0) = q_0(x), \quad 0 < x < \infty,$$

$$q(0,t) = f_0(t) \text{ and } q_x(0,t) = f_1(t), \ \textbf{or}\ q(0,t) = f_0(t) \text{ and } q_{xx}(0,t) = f_2(t),$$

$$\textbf{or}\ q_x(0,t) = f_1(t) \text{ and } q_{xx}(0,t) = f_2(t), \quad t > 0,$$

where $\{f_j(t)\}_0^2$ and $q_0(x)$ are smooth functions, $q_0(x)$ has sufficient decay as $x \to \infty$, and the given functions are compatible at $x = t = 0$.

For each of these problems use the transform pair defined in (a) to express $q(x,t)$ as an integral in the complex k-plane, uniquely defined in terms of appropriate transforms of the given initial and boundary conditions.

2 Assume that there exists a real-valued function $q(x, y)$ which satisfies the Laplace equation in the semi-strip $0 < x < \infty, \quad 0 < y < l$, $l > 0$, which decays as $x \to \infty$ for all $0 < y < l$, and which has sufficient smoothness all the way to the boundary.

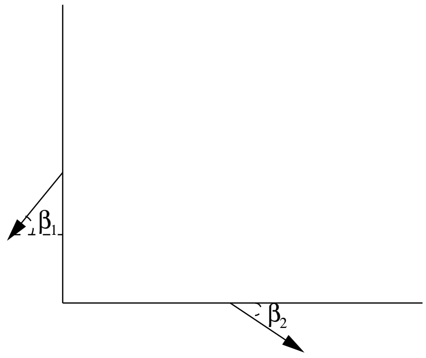
(a) By performing the spectral analysis of the differential form

$$d[\mu(z, k)e^{-ikz}] = e^{-ikz} q_z, \quad z = x + iy, \quad 0 < x < \infty, \quad 0 < y < l,$$

where $k \in \mathbf{C}$, find an integral representation for q_z in terms of appropriate transforms of the Dirichlet and of the Neumann boundary values.

(b) By analysing the associated global relation, compute the sine transforms of the Neumann boundary values $q_y(x, 0)$ and $q_y(x, l)$, in terms of the Dirichlet boundary values $q(x, 0), q(x, l), q(0, y)$.

3 Assume that there exists a real-valued function $q(x,y)$ which satisfies the Laplace equation in the quarter plane $0 < x < \infty, 0 < y < \infty$, which decays for large x and large y, and which has sufficient smoothness all the way to the boundary. Suppose that the derivative of the function q is prescribed along the direction making an angle β_1 with the y-axis as well as along the direction making an angle β_2 with the x-axis, see below,



i.e,

$$-q_x(0,y)\sin\beta_1 + q_y(0,y)\cos\beta_1 = h_1(y), \quad 0 < y < \infty,$$
$$-q_y(x,0)\sin\beta_2 + q_x(x,0)\cos\beta_2 = h_2(x), \quad 0 < x < \infty,$$

where h_1 and h_2 are given functions with appropriate smoothness and decay, $h_1(0) = h_2(0)$, and $\beta_1 + \beta_2$ takes the values 0 or $\pi/2$, or π.

(a) Let $j_1(y)$ and $j_2(x)$ denote the unknown derivatives in directions normal to the directions of the given derivatives, i.e,

$$-q_y(0,y)\sin\beta_1 - q_x(0,y)\cos\beta_1 = j_1(y), \quad 0 < y < \infty,$$
$$q_x(x,0)\sin\beta_2 + q_y(x,0)\cos\beta_2 = j_2(x), \quad 0 < x < \infty.$$

By expressing the unknown boundary values $q_y(0,y), q_x(0,y), q_y(x,0)$ and $q_x(x,0)$ in terms of the functions $h_1(y), h_2(x), j_1(y), j_2(x)$, compute the spectral functions

$$\widehat{q}_1(k) = \int_0^{i\infty} e^{-ikz} q_z dz, \quad \widehat{q}_2(k) = -\int_0^{\infty} e^{-ikz} q_z dz.$$

(b) Use the global relation and its consequences to express the spectral functions in terms of known functions, as well as a single unknown function which is bounded and analytic for $\mathrm{Im}k > 0$.

(c) Use the integral representation

$$q_z = \tfrac{1}{2\pi}\int_0^{\infty} e^{ikz}\widehat{q}_2(k)dk + \tfrac{1}{2\pi}\int_0^{i\infty} e^{ikz}\widehat{q}_1(k)dk,$$

to express q_z only in terms of $h_1(y)$ and $h_2(x)$.

END OF PAPER