UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 9 June, 2006 9 to 12

# PAPER 85

# BIFURCATIONS AND INSTABILITIES IN DISSIPATIVE SYSTEMS

*Attempt* ***THREE*** *questions.*

*There are* ***FOUR*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** A transition between modes of plasma confinement in a tokamak is described by the following differential equations for real variables $u(t)$ and $v(t)$:

$$\begin{aligned} \dot{u} &= \mu u + (1-\sigma)v - \frac{1}{2}(u-v)(u^2+v^2), \\ \dot{v} &= \mu v + (1+\sigma)u - \frac{1}{2}(u+v)(u^2+v^2), \end{aligned} \tag{1}$$

for real parameters $\mu$ and $\sigma$.

(a) Show that equilibrium points exist at $u = v = 0$ and when

$$u^2 + v^2 = \mu + \sigma \pm \sqrt{2 - (\mu - \sigma)^2}.$$

Identify all the local bifurcations from the equilibrium point $u = v = 0$. Show, using Dulac's criterion or otherwise, that the Hopf bifurcation is supercritical.

(b) Sketch the regions of the $(\sigma, \mu)$ plane in which (i) the origin is stable, and (ii) nontrivial equilibria exist. Briefly describe the codimension-two point $\sigma = -\mu = 1/\sqrt{2}$.

(c) Show that the point $\sigma = 1$, $\mu = 0$ is also a codimension-two point. Apply the rescalings $u = \varepsilon^2 \tilde{u}$, $v = \varepsilon \tilde{v}$, $\tilde{t} = \varepsilon t$, $\mu = \varepsilon^2 \tilde{\mu}$, $\sigma = 1 + \varepsilon^2 s$ to the equations (1). Dropping the $\tilde{}$, show that, for $\varepsilon = 0$, the rescaled equations have a conserved quantity $H = u^2 + \frac{s}{2}v^2 - \frac{1}{8}v^4$. Sketch the phase portrait when $\varepsilon = 0$ and $s > 0$. Give the value of $H$ that corresponds to the heteroclinic orbits.

(d) Explain how the rescaled equations are of use in detecting global bifurcations near $\sigma = 1$, $\mu = 0$. You should explain the significance of the expression

$$M(\mu, s) = \int_{-\sqrt{2s}}^{\sqrt{2s}} \left(\mu - \frac{v^2}{2}\right)\left(s + \frac{v^2}{2}\right) \,\mathrm{d}v,$$

but need not evaluate it.

**2** (a) Find the fixed points for the Hénon map

$$\begin{aligned} x_{n+1} &= y_n, \\ y_{n+1} &= \mu - b x_n - y_n^2, \end{aligned} \tag{2}$$

where $b > -1$. Investigate the regions of the $(b, \mu)$ plane in which the fixed points (i) exist and (ii) are stable.

Identify the bifurcations on the boundaries of these regions. Hence sketch a (partial) bifurcation diagram in the $(\mu, y)$ plane for $b = 1$.

Explain why the addition of small perturbation terms to (2) does not qualitatively change the bifurcation behaviour.

(b) Consider the 2D map

$$\begin{aligned} x_{n+1} &= (a+1)x_n - x_n^3 + y_n, \\ y_{n+1} &= (b-1)y_n - (b-1)x_n^3 + y_n^3, \end{aligned} \tag{3}$$

where $a$ and $b$ are real parameters. Investigate the codimension-one bifurcations from the fixed point $x = y = 0$. By locating bifurcating orbits at small amplitude, determine carefully whether the bifurcations are subcritical or supercritical. Indicate the location of bifurcating orbits on a sketch of the $(a, b)$ plane.

Describe the codimension-two bifurcation at $a = -1$, $b = 0$.

**3** Consider a 2D flow containing the following codimension-two bifurcation involving a local and a global bifurcation. Near $x = y = 0$ the flow can be well approximated by the equations

$$\dot{x} = x^2 - \mu\,,$$
$$\dot{y} = -\lambda y\,,$$

where $\lambda > 0$ is constant, and $\mu$ is a bifurcation parameter. Suppose that the half of the unstable manifold of $(\sqrt{\mu}, 0)$ that extends into $x > \sqrt{\mu}$ for small positive $\mu$ intersects the line $y = h$ 'downwards' at the point $x = \nu$, where $\nu$ may be greater than or less than $\sqrt{\mu}$.

(a) Explain why a global bifurcation occurs when $\nu = \sqrt{\mu}$. Assuming that $\mu > 0$ and $\sqrt{\mu} \ll h$, use $\int dx/\dot{x} = \int dy/\dot{y}$ to construct a 1D return map for points $(x, h)$ where $x > \sqrt{\mu}$ that describes the dynamics near the global bifurcation.

Show graphically that the return map has a stable fixed point when $\nu > \sqrt{\mu}$.

(b) By a similar construction show that a stable periodic orbit exists for the flow when $\mu < 0$, for all $\nu$.

(c) Sketch the phase portrait near the origin in the $(x, y)$ plane when $\nu < 0$, for small negative $\mu$. By integrating from $x = -h$ to $x = h$ show that the period $T$ of the periodic orbit varies as $T \sim \pi/\sqrt{-\mu}$ as $\mu \to 0^-$.

(d) Sketch the phase portraits near $\mu = \nu = 0$ in the three regions $\mu > \nu^2$, $0 < \mu < \nu^2$ and $\mu < 0$ for $\nu$ both positive and negative.

**4** (a) A mildly subcritical pattern forming instability in a domain $0 \leqslant x \leqslant L$ with periodic boundary conditions can be described by the PDE

$$w_t = [r - (1 + \partial_{xx}^2)^2]w + sw^3 - w^5. \tag{4}$$

Derive the subcritical cubic-quintic Ginzburg–Landau equation

$$A_T = \mu A + 3\hat{s}A|A|^2 - 10A|A|^4 + 4A_{XX} \tag{5}$$

for the evolution of small amplitude solutions to (4), by expanding $w(x,t) = \varepsilon w_1(x,t) + \varepsilon^2 w_2(x,t) + \cdots$ and introducing rescaled parameters $\mu$ and $\hat{s}$ defined by $r = \varepsilon^4 \mu$ and $s = \varepsilon^2 \hat{s}$. Justify the scalings you use for the slow space and time variables $X$ and $T$. State the solvability condition you use clearly.

(b) Sketch a bifurcation diagram in the $(\mu, A_0^2)$ plane showing the amplitude $A_0$ of real $X$-independent solutions of (5). Locate the saddle-node bifurcation point.

(c) By writing $A(X,T) = A_0 + a(T)e^{i\ell X}$, show that bifurcations to modulated ($X$-dependent) states with wavenumber $\ell > 0$ occur when

$$\mu - 4\ell^2 + 9\hat{s}A_0^2 - 50A_0^4 = 0.$$

Hence show that no real $X$-independent solution of (5) undergoes such a bifurcation if

$$L < \frac{8\pi\sqrt{10}}{3s}.$$

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