UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 12 June, 2006 1.30 to 3.30

PAPER 80

MACROSCOPIC BEHAVIOUR OF MICROSCOPIC STRUCTURES IN FLUID AND SOLID MEDIA

*Attempt **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) A laminate is composed of 2 different materials with isotropic conductivities $a_1\mathbf{I}$ and $a_2\mathbf{I}$. The interfaces between these two materials have normal $\mathbf{n} = (1, 0, 0)$. The volume fractions of these two materials in any samples of macroscopic dimensions are p_1 and p_2 respectively. Consider the steady state heat equation

$$-\boldsymbol{\nabla}.(\mathbf{a}(\mathbf{x})\boldsymbol{\nabla}u(\mathbf{x})) = 0,$$

where the conductivity $\mathbf{a}(\mathbf{x})$ equals $a_1\mathbf{I}$ in phase 1 and $a_2\mathbf{I}$ in phase 2. The temperature and the flux across the phase interfaces are continuous. Assuming that the temperature gradient $\boldsymbol{\nabla}u(\mathbf{x})$ is constant in each phase, deduce that the effective conductivity is given by

$$\begin{pmatrix} \dfrac{a_1a_2}{p_1a_2 + p_2a_1} & 0 & 0 \\ 0 & p_1a_1 + p_2a_2 & 0 \\ 0 & 0 & p_1a_1 + p_2a_2 \end{pmatrix}.$$

(ii) A composite is composed of a number of different components which are distributed periodically with the period being εY where ε is a micro lengthscale and Y is the unit cube in $\mathbb{R}^d$ ($d = 1, 2$ or 3). The conductivity of the composite $\mathbf{a}^\varepsilon(\mathbf{x})$ is given by $\mathbf{a}^\varepsilon(\mathbf{x}) = \mathbf{a}(\mathbf{x}/\varepsilon)$ where $\mathbf{a}(\mathbf{y}) = \{a_{ij}(\mathbf{y})\}$ is a Y-periodic matrix function that is positive definite for each $\mathbf{y} \in Y$. Consider the heat equation inside a d dimensional finite body Ω

$$-\frac{\partial}{\partial x_i}(a^\varepsilon_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}u^\varepsilon(\mathbf{x})) = f(\mathbf{x}),$$

with the zero temperature condition prescribed on the boundary of Ω. Using the two scale asymptotic expansion

$$u^\varepsilon(\mathbf{x}) = u_0(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \varepsilon u_1(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \varepsilon^2 u_2(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \dots,$$

where $u_i(\mathbf{x}, \mathbf{y})$ ($\mathbf{x} \in \Omega,\ \mathbf{y} \in Y$) is Y-periodic with respect to $\mathbf{y}$, deduce that the first term u_0 does not depend on $\mathbf{y}$ and satisfies the effective equation

$$-a^*_{ij}\frac{\partial^2 u_0}{\partial x_i \partial x_j} = f(\mathbf{x}).$$

Prove that the effective conductivity $\mathbf{a}^* = \{a^*_{ij}\}$ is given by

$$a^*_{ij} = \int_Y \left(a_{ij}(\mathbf{y}) + a_{ik}(\mathbf{y})\frac{\partial w_j(\mathbf{y})}{\partial y_k} \right) d\mathbf{y};$$

the functions $w_j(\mathbf{y})$ ($j = 1, \dots, d$) are Y-periodic and satisfy the equations

$$\frac{\partial}{\partial y_i}\left(a_{ik}(\mathbf{y})\frac{\partial w_j(\mathbf{y})}{\partial y_k} \right) = -\frac{\partial a_{ij}(\mathbf{y})}{\partial y_i}.$$

(iii) Consider the problem in part (ii) of this question in a one dimensional finite domain. The conductivity is now a scalar quantity $a^\varepsilon(x) = a(x/\varepsilon)$ where $a(y)$ is periodic with period 1. It is given that $a(y) = a_1$ when $0 \le y < p_1$ and $a(y) = a_2$ when $p_1 \le y < 1$ ($0 < p_1 < 1$, $p_1 + p_2 = 1$). Prove from the result in (ii) that the effective conductivity is $(a_1a_2)/(p_1a_2 + p_2a_1)$. Show that this is consistent with the result in part (i) of this question.

2 (i) A sphere B of radius r_1 made of a material of isotropic conductivity $a_1\mathbf{I}$ is placed inside an infinite medium of isotropic conductivity $a_0\mathbf{I}$. Consider the steady state heat equation

$$-\boldsymbol{\nabla}.(\mathbf{a}(\mathbf{x})\boldsymbol{\nabla}u(\mathbf{x})) = 0 \tag{1}$$

with the condition that $u(\mathbf{x}) \to x_3$ when $|\mathbf{x}| \to \infty$. The conductivity $\mathbf{a}(\mathbf{x})$ equals $a_1\mathbf{I}$ when $\mathbf{x}$ is inside the sphere and $a_0\mathbf{I}$ when $\mathbf{x}$ is outside. The centre of the sphere is at the origin. By considering the special form of the solution $u(\mathbf{x}) = v(r)\cos\theta$ in the spherical coordinates (r, θ, ϕ) $(x_3 = r\cos\theta)$, prove that

$$\begin{aligned} u(\mathbf{x}) &= x_3 - \frac{r_1^3(a_1 - a_0)}{r^3(a_1 + 2a_0)}x_3, \quad \mathbf{x} \notin B, \\ &= \frac{3a_0}{a_1 + 2a_0}x_3, \quad \mathbf{x} \in B, \end{aligned}$$

where $r = |\mathbf{x}|$. Deduce the polarization field $\mathbf{P}(\mathbf{x})$. [*You may use the following form for the Laplacian in the spherical coordinates*

$$\nabla^2 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}(u_{\theta\theta} + \cot\theta u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}). \text{]}$$

(ii) A three dimensional body Ω contains N very small identical spheres $B_1, \ldots, B_N$ which are well separated from each other. The conductivity of the inclusions is $a_1\mathbf{I}$ and the conductivity outside is $a_0\mathbf{I}$. By using the polarization field $\mathbf{P}(\mathbf{x})$ in part (i) of this question, deduce that to the first order of the volume fraction p of all the inclusions in Ω, the effective conductivity $\mathbf{a}^*$ is approximated by

$$\mathbf{a}^* = a_0\mathbf{I} + \frac{3pa_0(a_1 - a_0)}{a_1 + 2a_0}\mathbf{I}.$$

(iii) Assume that the body Ω is the unit cube i.e. $\Omega = [0,1]^3$. Consider a temperature field of the form $u(\mathbf{x}) = x_3 + v(\mathbf{x})$ where $v(\mathbf{x})$ is an Ω-periodic function. The mean temperature gradient in Ω is then $\mathbf{i} = (0, 0, 1)$. For a first approximation, we can write the polarization field as

$$\mathbf{P}(\mathbf{x}) = \sum_{i=1}^{N} \frac{3a_0(a_1 - a_0)}{a_1 + 2a_0}\chi_i(\mathbf{x})\mathbf{i},$$

where $\chi_i(\mathbf{x})$ is the characteristic function of the ith inclusion B_i. Consider the perturbation field $\mathbf{e}(\mathbf{x}) = \boldsymbol{\nabla}u(\mathbf{x}) - \mathbf{i}$, so that $\mathbf{e} = \boldsymbol{\nabla}u'(\mathbf{x})$ for some Ω-periodic function $u'(\mathbf{x})$ that satisfies

$$\nabla^2 u'(\mathbf{x}) = -\frac{1}{a_0}\boldsymbol{\nabla}.\mathbf{P}(\mathbf{x}).$$

This equation is then solved by the Fourier transform (you are not asked to do this). For an inclusion B_i with centre $\mathbf{y}$, the field $\mathbf{e}(\mathbf{x})$ on the surface of a sphere S which is significantly larger than B_i with the same centre $\mathbf{y}$ (all the other inclusions B_j $j \neq i$ are outside S) is then given by

$$\mathbf{e}(\mathbf{x}) = -\frac{3a_0(a_1 - a_0)}{(a_1 + 2a_0)}\boldsymbol{\Lambda}_i\mathbf{i}/p_i,$$

where $p_i = |B_i|/|\Omega|$ is the volume fraction of the inclusion B_i and $\mathbf{\Lambda}_i$ is a 3×3 matrix. From this deduce the following improved estimate for the effective conductivity

$$\mathbf{a}^* = a_0\mathbf{I} + \frac{3pa_0(a_1 - a_0)}{a_1 + 2a_0}\mathbf{I} - \left(\frac{3a_0(a_1 - a_0)}{a_1 + 2a_0}\right)^2 \sum_{i=1}^{N} \mathbf{\Lambda}_i.$$

(iv) Now assume that Ω is a sphere of radius R_0. The effective conductivity of this sphere is assumed to be isotropic and can be written as $a^*\mathbf{I}$. In part (i), without the sphere B, the solution of equation (1) is given by $u(\mathbf{x}) = x_3$ so at a point $\mathbf{x}$ which is far from the origin, this solution is perturbed by a term $-(r_1^3(a_1 - a_0))/(r^3(a_1 + 2a_0))x_3$ due to the sphere B. The spheres $B_1, \ldots, B_N$ are well separated so that their total effect on the solution at this point can be approximated as the summation of the effect of each individual sphere. Use this to deduce the Maxwell approximation

$$a^* = a_0 + \frac{3pa_0(a_1 - a_0)}{3a_0 + (1 - p)(a_1 - a_0)}.$$

When the composite is isotropic, the matrices $\mathbf{\Lambda}_i$ in (iii) satisfy $\sum_{i=1}^{N} \mathbf{\Lambda}_i \approx -(p^2/(3a_0))\mathbf{I}$. From this show that the Maxwell approximation is consistent with the approximation in (iii).

3 (i) Consider the heat equation

$$-\boldsymbol{\nabla}.(\mathbf{a}(\mathbf{x})\boldsymbol{\nabla}u(\mathbf{x})) = 0 \text{ in } \Omega,$$

$$u(\mathbf{x}) = \boldsymbol{\lambda}.\mathbf{x} \text{ on } \partial\Omega,$$

in a three dimensional finite body Ω where the conductivity $\mathbf{a}(\mathbf{x})$ is symmetric. For each function $v(\mathbf{x})$, let

$$J(v) = \int_\Omega \mathbf{a}(\mathbf{x})\boldsymbol{\nabla}v(\mathbf{x}).\boldsymbol{\nabla}v(\mathbf{x})d\mathbf{x}.$$

Prove the minimum energy principle

$$J(u) = \min\{J(v) : \; v(\mathbf{x}) = \boldsymbol{\lambda}.\mathbf{x} \text{ on } \partial\Omega\}.$$

(ii) Assume that Ω is a composite consisting of two phases with isotropic conductivities $a_1\mathbf{I}$ and $a_2\mathbf{I}$ ($a_1 < a_2$). The volume fractions of the phases are p_1 and p_2 respectively. The effective conductivity is isotropic and is denoted by $a^*\mathbf{I}$. Deduce the Voigt bound $a^* \le p_1a_1 + p_2a_2$.

(iii) The Hashin-Shtrikman variational principle for the material in part (ii) can be written as

$$J(u) \le |\Omega|a_2\boldsymbol{\lambda}.\boldsymbol{\lambda} + 2\boldsymbol{\lambda}.\int_\Omega \mathbf{q}(\mathbf{x})d\mathbf{x} + \int_\Omega \mathbf{q}(\mathbf{x}).\boldsymbol{\nabla}v(\mathbf{x})d\mathbf{x} - \int_\Omega (\mathbf{a} - a_2\mathbf{I})^{-1}\mathbf{q}(\mathbf{x}).\mathbf{q}(\mathbf{x})d\mathbf{x}, \quad (2)$$

where $\mathbf{q}(\mathbf{x})$ is a vector field and $v(\mathbf{x})$ satisfies

$$a_2\nabla^2 v(\mathbf{x}) + \boldsymbol{\nabla}.\mathbf{q}(\mathbf{x}) = 0 \text{ in } \Omega,$$
$$v(\mathbf{x}) = 0 \text{ on } \partial\Omega.$$

The solution of this equation is

$$v(\mathbf{x}) = -\int_\Omega \mathbf{q}(\mathbf{x}').\boldsymbol{\nabla}_{\mathbf{x}'}g(\mathbf{x},\mathbf{x}')d\mathbf{x}',$$

where $g(\mathbf{x},\mathbf{x}')$ is the Green function that satisfies

$$a_2\nabla^2_{\mathbf{x}'}g(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}'),$$
$$g(\mathbf{x},\mathbf{x}') = 0 \text{ when } \mathbf{x}' \in \partial\Omega$$

The probability that a point $\mathbf{x}$ is in phase 1 is p_1 and in phase 2 is p_2. The material is isotropic so that the probability that $\mathbf{x}$ is in phase i and $\mathbf{x}'$ is in phase j is $p_{ij}(|\mathbf{x}-\mathbf{x}'|)$ for some function $p_{ij}(r)$ $(i,j = 1,2)$. When $|\mathbf{x}-\mathbf{x}'| > \rho$ for some microscopic length ρ, $p_{ij}(|\mathbf{x}-\mathbf{x}'|) = p_ip_j$. Furthermore when $|\mathbf{x}-\mathbf{x}'| \le \rho$,

$$\boldsymbol{\nabla}_{\mathbf{x}}\boldsymbol{\nabla}_{\mathbf{x}'}g(x,x') \approx \frac{1}{3a_2}\mathbf{I}\delta(\mathbf{x}-\mathbf{x}') - \mathbf{H}(\mathbf{x}-\mathbf{x}'),$$

where the matrix function $\mathbf{H}(\mathbf{x})$ satisfies: for all $R > 0$

$$\int_{\partial B_R} \mathbf{H}(\mathbf{x}) dS = \mathbf{0},$$

where B_R is the ball of radius R with centre at the origin. Moreover

$$\int_\Omega \int_\Omega \boldsymbol{\nabla}_{\mathbf{x}} \boldsymbol{\nabla}_{\mathbf{x}'} g(\mathbf{x}, \mathbf{x}') d\mathbf{x} d\mathbf{x}' = 0.$$

Choose the vector field $\mathbf{q}(\mathbf{x})$ as $t(a_1 - a_2)\boldsymbol{\lambda}\chi_1(\mathbf{x})$ where $\chi_1(\mathbf{x})$ is the characteristic function of phase 1. Taking the ensemble average of the right hand side of (2), prove that

$$a^* \leq (a_1 - a_2)p_1 \left(2t - t^2 \left(\frac{a_1 - a_2}{3a_2} p_2 + 1\right)\right) + a_2.$$

From this deduce the Hashin-Shtrikman upper bound

$$a^* \leq a_2 + \frac{3a_2(a_1 - a_2)p_1}{3a_2 + p_2(a_1 - a_2)}.$$

(iv) Without proof, write down the Hashin-Shtrikman variational principle for the lower bound and the Hashin-Shtrikman lower bound.

END OF PAPER