UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 8 June, 2006 1.30 to 4.30

PAPER 8

SEMIGROUPS OF OPERATORS

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Suppose that E is a complex Banach space. What is a C_0 *semigroup* $(T_t)_{t\geqslant 0}$ acting on E? What is its *infinitesimal generator* Z? Explain the sense in which T_t and Z commute.

What is a *contraction semigroup*? Suppose that $(T_t)_{t\geqslant 0}$ is a contraction semigroup acting on E. Show that if $\Re(\lambda) > 0$ then there exists $L_\lambda \in L(E)$ such that $L_\lambda(E) \subseteq D(Z)$ and $(\lambda I - Z)L_\lambda(f) = f$ for all $f \in E$.

Show that Z is a closed linear operator on E.

2 What is a *dissipative* operator?

Suppose that Z is a closed linear operator on a Hilbert space H, with dense domain. Show that Z is dissipative if and only if $\Re\langle Z(f), f\rangle \leqslant 0$ for all $f \in D(Z)$. Show that $\alpha > 0$ is in the spectrum of Z if and only if α is an eigenvalue of Z^*.

Let $H = l_2(Z^+)$, let $Q(f)_0 = -f_0$, let $Q(f)_n = 2^{2n-1}f_{n-1} - 2^{2n}f_n$ for $n > 0$, and let $D(Q) = \{f \in H : Q(f) \in H\}$. Show that Q is dissipative. Does Q generate a contraction semigroup?

3 Let $H = L^2([0,1],\mu)$, where μ is Lebesgue measure, and let C^1 be the space of continuous functions on $[0,1]$ with continuous derivative. If $f \in H$ let $J(f)(x) = \int_0^x f(t)\,dt$. Show that $J \in L(H)$ and that J is one-one.

Let $D(A) = J(H)$, and if $f \in D(A)$ let $A(f) = iJ^{-1}(f)$, so that A is a closed operator. Show that A has a dense domain. Determine the spectrum of A. *[You may assume that the spectrum $\sigma(J)$ of J is $\{0\}$.]*

Let $H_0 = \{f : J(f)(1) = 0\}$, let $D(A_0) = J(H_0)$, and let A_0 be the restriction of A to A_0. Show that A_0 is closed and symmetric.

Show that $C^1 \subseteq D(A_0^*)$. What are the eigenvalues of A_0^*? What is the spectrum of A_0?

4 Suppose that $(P_t)_{t>0}$ is a symmetric Feller semigroup, with invariant probability measure μ.

Show that if f is positive and $q > 2$ then the joint energy satisfies

$$\mathcal{E}_\mu(f^{q/2}) \leqslant \frac{q^2}{4(q-1)}\mathcal{E}_\mu(f^{q-1}, f).$$

Suppose that μ satisfies a logarithmic Sobolev inequality with constant c_{LS}. Let $q(t) = 1 + e^{4t/c_{LS}}$. Show that if $f \in L^2(\mu)$ then $P_t(f) \in L^{q(t)}(\mu)$, and $\|P_t(f)\|_{q(t)} \leqslant \|f\|_2$.

5 What are the *creation* and *annihilation operators* a^+ and a^- on $L^2(R, \gamma_1)$?

Let $(P_t)_{t \geqslant 0}$ be the Ornstein-Uhlenbeck semigroup generated by $L = -a^+a^-$. Calculate the squared gradient operator $\Gamma(f, g)$ acting on functions f, g in a standard algebra A, and calculate the joint energy $\mathcal{E}_{\gamma_1}(f, g)$.

Show that γ_1 has logarithmic Sobolev constant equal to 2.

[You may assume that if $f > 0$ then $(P_t(g))^2 \leqslant P_t(g^2/f)P_t(f)$.]

END OF PAPER