UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 12 June, 2006 9 to 11

PAPER 79

HYDRODYNAMIC TURBULENCE

Attempt ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
A data sheet of three pages
is attached.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) What is the zeroth law of turbulence? Let ν be the kinematic viscosity and ϵ the average dissipation of kinetic energy per unit mass. Estimate the magnitude of the Kolmogorov microscales, v and η, in terms of the integral scales, u and ℓ, and the Reynolds number $u\ell/\nu$. Find the equivalent expressions in terms of ν and ϵ.

(ii) Kolmogorov's 1941 theory assumes local isotropy in the universal equilibrium range. Explain what the terms 'local isotropy' and 'universal equilibrium range' mean in this context.

State Kolmogorov's first similarity hypothesis and show that it implies

$$\left\langle (\Delta \mathrm{v})^2 \right\rangle = \mathrm{v}^2 F(r/\eta) \qquad , \qquad r << \ell$$

for some universal function F, where $\Delta \mathrm{v}$ is the velocity increment. State the second similarity hypothesis and find the corresponding form of $\left\langle (\Delta \mathrm{v})^2 \right\rangle$ in the inertial subrange.

What, according to this theory, is the form of $\left\langle (\Delta \mathrm{v})^P \right\rangle$ in the same range?

(iii) Explain what is meant by external intermittency, internal intermittency at the equilibrium range, and internal intermittency at the integral scale. Why is integral-scale internal intermittency non-universal? What aspect of the 1941 theory did Landau criticise and why?

What is Kolmogorov's refined similarity hypothesis of 1962? Show that it demands that

$$\left\langle (\Delta \mathrm{v})^P \right\rangle = \beta_P \left\langle \epsilon_{AV}^{P/3}(r) \right\rangle r^{P/3},$$

where $\epsilon_{AV}(r)$ is the dissipation averaged over the scale r.

(iv) Discuss how the 1941 theory can be adapted to describe the mixing of a passive scalar and deduce the passive-scalar analogue of the $2/3^{rds}$ law.

2 (i) The third invariant of the velocity gradient tensor, $A_{ij} = \partial u_i/\partial x_j$, can be expressed in terms of the strain-rate tensor, S_{ij}, and vorticity $\boldsymbol{\omega}$ according to

$$R = A_{ij}A_{jk}A_{ki} = S_{ij}S_{jk}S_{ki} + \frac{3}{4}\omega_i\omega_j S_{ij}.$$

Confirm that R can be written as the divergence of

$$\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_k}u_k - \frac{1}{2}u_i\frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_k}$$

and hence show that, in homogeneous turbulence,

$$\langle S_{ij}S_{jk}S_{ki}\rangle = -\frac{3}{4}\langle\omega_i\omega_j S_{ij}\rangle\,.$$

(ii) Let a, b and c be the principal rates of strain at any location within the turbulence. Confirm that $a^3 + b^3 + c^3 = 3(abc)$ and hence show that

$$\langle abc\rangle = -\frac{1}{4}\langle\omega_i\omega_j S_{ij}\rangle$$

(iii) Explain why the result above suggests that bi-axial strain is more common than axial strain in turbulence and discuss the significance of this for the morphology of the vorticity field.

(iv) Burgers' vortex consists of a vortex tube $\omega_z = (\Gamma/\pi\delta^2)\exp[-r^2/\delta^2]$ sitting in the irrotational straining flow $u_r = -\frac{1}{2}\alpha r, \quad u_z = \alpha z$ in cylindrical polar coordinates (r, θ, z). The tube radius is $\delta = (4\nu/\alpha)^{1/2}$. Show that, in the limit of $\Gamma/\nu \to \infty$, the viscous dissipation per unit length of the tube is independent of viscosity ν. Explain why this makes Burgers' vortex a good candidate for the centres of dissipation in a turbulent flow.

3 (i) Starting with the Karman-Howarth equation,

$$\frac{\partial}{\partial t}\left[u^2 f\right] = \frac{1}{r^4}\frac{\partial}{\partial r}r^4\left[u^3 K + 2\nu\frac{\partial}{\partial r}\left(u^2 f\right)\right],$$

show that the second and third-order structure functions are related by Kolmogorov's equation in the universal equilibrium range:

$$\left\langle (\Delta \mathrm{v})^3 \right\rangle - 6\nu\frac{\partial}{\partial r}\left\langle (\Delta \mathrm{v})^2 \right\rangle = -\frac{4}{5}\epsilon r \qquad , \qquad r << \ell.$$

What is the $4/5^{th}$ law and why is it of particular importance for Kolmogorov's 1941 theory of the small scales?

(ii) Using the expansions

$$u^2 f = u^2 - \left\langle \boldsymbol{\omega}^2 \right\rangle \tfrac{r^2}{30} + \left\langle (\nabla \times \boldsymbol{\omega})^2 \right\rangle \tfrac{r^4}{840} + \cdots$$
$$\left\langle (\Delta \mathrm{v})^3 \right\rangle = \left\langle (\partial u_x/\partial x)^3 \right\rangle r^3 + \cdots$$

in conjunction with the Karman-Howarth equation, show that

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\left\langle \boldsymbol{\omega}^2 \right\rangle\right] = -\frac{35}{2}\left\langle (\partial u_x/\partial x)^3 \right\rangle - \nu\left\langle (\nabla \times \boldsymbol{\omega})^2 \right\rangle.$$

Hence confirm that the average rate of vortex stretching, $\langle \omega_i \omega_j S_{ij} \rangle$, is related to the skewness of $\partial u_x/\partial x$, S_0, by

$$\langle \omega_i \omega_j S_{ij} \rangle = -\frac{7}{6\sqrt{15}} S_0 \left\langle \boldsymbol{\omega}^2 \right\rangle^{3/2}.$$

Why does this tell us that turbulence is intrinsically non-Gaussian? Sketch the probability density function for $\partial u_x/\partial x$.

(iii) Show that Kolmogorov's $2/3^{rd}$ law demands that the skewness, $S(r)$, of the velocity increment, $\Delta \mathrm{v}(r)$, is constant across the inertial subrange and equal to

$$S(r) = -\frac{4/5}{\beta^{3/2}},$$

where β is Kolmogorov's constant. Consider the constant skewness closure model, in which S is taken as constant across the entire universal equilibrium range. Deduce the corresponding governing equation for normalised structure function

$$h = \frac{\left\langle (\Delta \mathrm{v})^2 \right\rangle}{\beta\left(15\beta\right)^{1/2}\mathrm{v}^2}$$

in terms of the normalised separation

$$x = \frac{r}{\left(15\beta\right)^{3/4}\eta}.$$

Discuss the relative merits of this closure model and the more common EDQNM closure scheme.

END OF PAPER