UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 2 June, 2006 9 to 12

PAPER 78

SLOW VISCOUS FLOW

*Attempt at most **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

Substantially complete answers will be viewed more favourably than fragments.

A mark of distinction standard can be gained by substantially complete answers to two questions.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

The Papkovich–Neuber representation of Stokes flow is

$$\mathbf{u} = \boldsymbol{\nabla}(\mathbf{x}\cdot\boldsymbol{\Phi} + \chi) - 2\boldsymbol{\Phi}, \quad p = 2\mu\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}, \quad \text{where } \nabla^2\chi = 0 \text{ and } \nabla^2\boldsymbol{\Phi} = \mathbf{0}.$$

1 Derive the Stokes flow (the Stokeslet solution) corresponding to a point force $\mathbf{F}\,\delta(\mathbf{x})$ in an unbounded fluid, and show that your solution has zero net mass flux from the origin. Write down the tensors $\mathbf{J}(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})$ that give the solution as $\mathbf{u}(\mathbf{x}) = \mathbf{F}\cdot\mathbf{J}(\mathbf{x})$ and $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{F}\cdot\mathbf{K}(\mathbf{x})$.

State the Reciprocal Theorem for two Stokes flows $\mathbf{u}_i$ $(i = 1, 2)$ with body forces $\mathbf{f}_i$ in a volume V with smooth surface ∂V and outward normal $\mathbf{n}$. For a Stokes flow in V with *no* body force, deduce the integral representation

$$\int_{\partial V} \mathbf{J}(\mathbf{y}-\mathbf{x})\cdot\boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}\,dS_{\mathbf{x}} + \int_{\partial V} \mathbf{u}(\mathbf{x})\cdot\mathbf{K}(\mathbf{y}-\mathbf{x})\cdot\mathbf{n}\,dS_{\mathbf{x}} = \begin{cases} \mathbf{u}(\mathbf{y}) & \mathbf{y}\in V \\ 0 & \mathbf{y}\notin V \\ \tfrac{1}{2}\mathbf{u}(\mathbf{y}) & \mathbf{y}\in\partial V, \end{cases} \qquad (*)$$

explaining the key steps in the derivation.

(a) Use $(*)$ to obtain an integral equation for the velocity $\mathbf{u}(\mathbf{y})$ at a point $\mathbf{y}$ on the interface ∂V of a drop of viscosity $\lambda\mu$ immersed in unbounded fluid of viscosity μ due to the action of a constant surface tension γ between the fluids.

(b) A flexible body with surface ∂V, outward normal $\mathbf{n}'$ and typical size a moves through unbounded fluid. The tensor constants $\boldsymbol{\Sigma}$, $\mathbf{U}$ and $\mathbf{M}$ are defined by

$$\Sigma_{\alpha\beta} = \int_{\partial V} x_\alpha \sigma_{\beta k} n'_k\,dS_{\mathbf{x}}, \quad U_{\alpha\beta} = \int_{\partial V} u_\alpha n'_\beta\,dS_{\mathbf{x}} \quad \text{and} \quad M_{\alpha\beta} = \Sigma_{\alpha\beta} - 2\mu U_{\alpha\beta}.$$

The origin is chosen to be inside the body so that $|\mathbf{x}| = O(a)$ on ∂V.

Use $(*)$ to show that for $r \gg a$, where $r = |\mathbf{y}|$,

$$\mathbf{u}(\mathbf{y}) = \mathbf{J}(\mathbf{y})\cdot\mathbf{F} + \frac{\mathbf{G}\wedge\mathbf{y}}{8\pi\mu r^3} + \frac{Q\mathbf{y}}{4\pi r^3} - \frac{3(\mathbf{y}\cdot\mathbf{S}\cdot\mathbf{y})\mathbf{y}}{8\pi\mu r^5} + O(r^{-3}),$$

where $G_j = -\epsilon_{j\alpha\beta}\Sigma_{\alpha\beta}$, $Q = \mathrm{Tr}(\mathbf{U})$, $\mathbf{S} = \frac{1}{2}(\mathbf{M}+\mathbf{M}^{\mathrm{T}}) - \frac{1}{3}\mathbf{I}\,\mathrm{Tr}(\mathbf{M})$, and $\mathbf{F}$ is to be indentified.

Give a physical interpretation of the constants $\mathbf{F}$, $\mathbf{G}$ and Q.

2 State and prove the Reciprocal Theorem for two Stokes flows in which the body forces are zero.

Prove that the resistance matrix, giving the force $\mathbf{F}$ and couple $\mathbf{G}$ exerted by a rigid body when moving with velocity $\mathbf{U}$ and angular velocity $\boldsymbol{\Omega}$ through surrounding viscous fluid, is both symmetric and positive definite.

The centreline of a helical wire of thickness ϵb, with $\epsilon \ll 1$, is given in Cartesian coordinates by

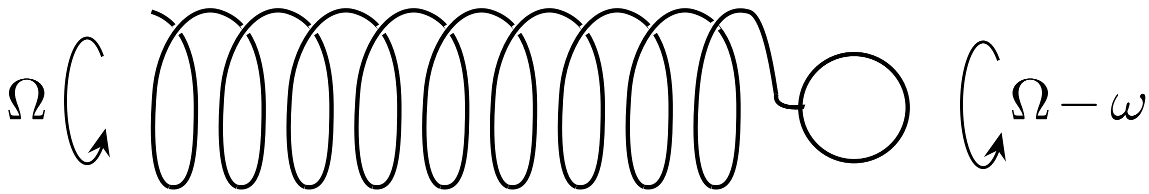
$$\mathbf{X}(\theta) = b(\cos\theta, \sin\theta, \theta\tan\phi), \qquad 0 \leqslant \theta \leqslant N\pi, \qquad (*)$$

where N is a large integer and ϕ is the pitch of the helix (i.e. the constant angle between $d\mathbf{X}/d\theta$ and the xy-plane). Show that the arc-length s is given by $ds = b\sec\phi\, d\theta$, so that the total length is $L = N\pi b\sec\phi$.

Find the force and the couple exerted by the helix in (i) pure translation in the z-direction with speed U and (ii) pure rotation about the z-axis with angular velocity Ω. (It is **not** necessary to show that the non-axial components are zero.)

[*You may assume the slender-body formula* $\mathbf{f}(\mathbf{X}) = C(\mathbf{I} - \frac{1}{2}\mathbf{X}'\mathbf{X}') \cdot \mathbf{V}(\mathbf{X})$, *where* $C = 4\pi\mu/|\ln\epsilon|$ *and* $\mathbf{X}' = d\mathbf{X}/ds$.]

A micro-organism has the same density as the surrounding fluid. It consists of a long helical flagellum of the shape given by $(*)$ attached to a small spherical head of radius $a \ll L$ in such a way that centre of the sphere lies on the axis of the helix. A 'molecular motor' rotates the flagellum about its axis with angular velocity ω relative to the head.



Because $a \ll L$, the head has a negligible effect on the total force and couple experienced by the flagellum. Moreover, the head lies within the large-scale flow generated by the flagellum and hence experiences only a couple $8\pi\mu a^3\omega$ due to the relative motion. Use these approximations to show that the axial velocity of the micro-organism is given by

$$U = \frac{\omega a^3 \sin 2\phi}{bL/|\ln\epsilon|}\ .$$

Comment on the predicted dependence of U on ϕ and on L, with all other parameters held fixed.

Using the same approximations, show that the rate of working by the organism is $8\pi\mu a^3\omega^2$, independent of the values of U and Ω.

3 The concentration C of surfactant on the surface of an inviscid bubble immersed in a very viscous fluid satisfies

$$\frac{DC}{Dt} = -C[\boldsymbol{\nabla}_s \cdot \mathbf{u}_s + (\mathbf{u}\cdot\mathbf{n})\boldsymbol{\nabla}_s \cdot \mathbf{n}] + D_s \nabla_s^2 C - k(C - C_0)\,, \tag{1}$$

where $\mathbf{n}$ is the unit normal out of the bubble; $\mathbf{u}_s = \mathbf{I}_s\cdot\mathbf{u}$, $\boldsymbol{\nabla}_s = \mathbf{I}_s\cdot\boldsymbol{\nabla}$ and $(\mathbf{I}_s)_{ij} = \delta_{ij} - n_i n_j$. Describe the physical interpretation of each of the terms in (1).

Assume that the steady concentration on a spherical bubble of radius a rising vertically with velocity $\mathbf{U}$ can be written as $C = C_0 + C'$, where $|C'| \ll C_0$ and C_0 is uniform. Derive an appropriately simplified form of (1) in the frame where the bubble is at rest. Explain why the velocity on the interface should be given by

$$\mathbf{u}(\mathbf{x}) = A\mathbf{I}_s(\mathbf{x})\cdot\mathbf{U}$$

for some constant A.

Show that $\boldsymbol{\nabla}_s\mathbf{n} = \mathbf{I}_s/a$ and derive expressions for $\boldsymbol{\nabla}_s \cdot \mathbf{n}$, $\nabla_s^2\mathbf{n}$ and $\boldsymbol{\nabla}_s \cdot \mathbf{I}_s \cdot \mathbf{U}$. Hence verify that

$$C' = B\mathbf{U}\cdot\mathbf{n}$$

and determine the constant B as a multiple of A. State conditions under which the assumption $|C'| \ll C_0$ is valid and give a physical interpretation of their meaning.

For $|C'| \ll C_0$ the surface-tension coefficient is given by $\gamma(C) = \gamma_0 - \gamma_1 C'$, where $\gamma_0 = \gamma(C_0)$ and γ_1 is a positive constant. Write down the general stress boundary condition for a fluid–fluid interface with surface tension γ and curvature κ, and show that in this case

$$[\mathbf{I}_s\cdot\boldsymbol{\sigma}\cdot\mathbf{n}]_-^+ = 6\mu A\lambda\,\mathbf{I}_s\cdot\mathbf{U}/a\ ,$$

where λ is a constant that should be identified.

Assuming that $\mathbf{u} \to -\mathbf{U}$ as $r/a \to \infty$, explain why the Papkovich–Neuber potentials for the flow can be written in the form

$$\boldsymbol{\Phi} = \mathbf{U} + \alpha a\,\mathbf{U}\,\frac{1}{r}\ , \qquad \chi = \beta a^3\,\mathbf{U}\cdot\boldsymbol{\nabla}\,\frac{1}{r}\,,$$

where α and β are constants. These potentials correspond to

$$\mathbf{u} = -(1+\alpha+\beta)\mathbf{U} + (3\beta-\alpha)(\mathbf{U}\cdot\mathbf{n})\mathbf{n} \tag{2}$$

$$\boldsymbol{\sigma}\cdot\mathbf{n} = \frac{12\mu}{a}\left\{\,\beta\mathbf{U} + (\alpha-3\beta)(\mathbf{U}\cdot\mathbf{n})\mathbf{n}\,\right\} \tag{3}$$

on $r = a$. Use (2) and (3) to determine α, β and A in terms of λ.

Interpret the limits $\lambda \to 0$ and $\lambda \to \infty$ in terms of the surfactant effects on the tangential velocity and stress on the interface.

What balances the value of $\mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}$ given by (3)?

4 A thin film of viscous fluid flows with a typical velocity U between two rigid surfaces with a typical gap width H that varies on a lengthscale $L \gg H$. Use scaling arguments to show that the typical shear stress τ and the typical pressure variations P satisfy $\tau/P \ll 1$.

The gap between two rigid spheres of radius a and $a + \Delta$, with $0 < \Delta \ll a$, is filled with fluid of viscosity μ. The outer sphere is stationary. Assume that the lubrication equations describe the flow produced by any motion of the inner sphere.

The centre of the inner sphere is a distance $\alpha\Delta$ (with $-1 < \alpha < 1$) directly below that of the outer sphere. Show that the gap thickness $h(\theta)$ between the spheres is approximately given by

$$h = (1 - \alpha\cos\theta)\Delta,$$

where θ is the angle to the downward vertical.

(i) Show that when the inner sphere moves downwards with speed $V = (d\alpha/dt)\Delta$ the fluid pressure is given by

$$p(\theta) = \frac{3\mu a^2 V}{\alpha\Delta^3(1 - \alpha\cos\theta)^2} + \text{const.}$$

Hence calculate the vertical force F acting on the inner sphere.

[*You may assume that* $\displaystyle\int_{-b}^{b} \frac{t\,dt}{(1-t)^2} = \frac{2b}{(1-b)^2} + \ln\left(\frac{1-b}{1+b}\right)$ for $|b| < 1$.]

(ii) Calculate also the couple G acting on the inner sphere when it rotates with angular velocity Ω about the vertical axis.

[*You may assume that* $\displaystyle\int_{-b}^{b} \frac{(a^2 - t^2)\,dt}{1-t} = 2b + (1 - a^2)\ln\left(\frac{1-b}{1+b}\right)$ for $|b| < 1$.]

(iii) The inner sphere now rotates with angular velocity Ω' about a horizontal axis, with its centre held stationary and with $0 < 1 - \alpha \ll 1$. Let $\epsilon = 1 - \alpha$.

Use scaling arguments to show that the magnitude of the horizontal couple G' exerted by the inner sphere about its centre is $O(\mu\Omega' a^4/\Delta)$. You should consider separately the contributions from the regions where $h = O(\Delta)$ and $h = O(\epsilon\Delta)$ and show that they have comparable magnitude.

Estimate also the magnitude of the horizontal force F' required to hold the centre stationary and explain why the vertical force required is zero.

END OF PAPER