UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 2 June 2006 1.30 to 4.30

PAPER 77

FUNDAMENTALS OF ATMOSPHERE–OCEAN DYNAMICS

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

Clarity and explicitness of reasoning will attract more credit than perfection of computational detail.

(x, y, z) *denotes right-handed Cartesian coordinates and* (u, v, w) *the corresponding velocity components;* t *is time; the gravitational acceleration is* $(0, 0, -g)$ *where* g *is a positive constant;* $\hat{z} = (0, 0, 1)$ *is a unit vector directed vertically upward.*
The fluid is always incompressible. 'Ideal fluid' always means that buoyancy diffusion can be neglected where relevant, as well as viscosity. N *denotes the buoyancy frequency of a stratified fluid.*

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write down the nonlinear Boussinesq momentum, mass-continuity and buoyancy equations for a continuously stratified, non-rotating ideal fluid. Buoyancy effects should be represented in the equations by a suitably defined buoyancy acceleration $\sigma(\mathbf{x}, t)$, to be specified. Define the buoyancy frequency $N(z)$. State briefly the conditions under which the Boussinesq approximation is valid, given the incompressibility of the fluid.

In a fluid that is unbounded above, 2-dimensional internal gravity waves of small amplitude are generated by a moving boundary $z = \epsilon \sin(kx - \omega t)$ where ϵ, k and ω are real constants with $|\omega| < N$. Carefully state, and justify, the boundary condition that can be applied at $z = 0$, when ϵ is sufficiently small in a sense to be specified.

When N is constant, derive the dispersion relation for a plane internal gravity wave, by using appropriately tilted axes or otherwise. State the appropriate radiation condition for the wave generation problem. Derive the steady-state solution to that problem, satisfying the radiation condition as well as the boundary condition at $z = 0$. Show by a sketch how the wavecrests are oriented when the boundary undulations move from left to right. Give an expression for the angle the wavecrests make with the horizontal. [*You may quote without proof any standard results on group velocity.*]

Find the time-averaged rate W at which the boundary does work upon the fluid above it, per unit horizontal area.

Show from the equations of motion that, provided the pressure and velocity fields satisfy certain conditions, which you should specify, then

$$\frac{\partial \bar{u}}{\partial t} = -\frac{\partial}{\partial z}\,\overline{uw}$$

where the overbars denote the average with respect to x.

In the foregoing wave generation problem, ϵ is now made a function of t that vanishes for $t < 0$ but grows slowly toward a small constant value ϵ_∞ as $t \to \infty$. Briefly state how to modify your solution, on the assumption that ϵ grows sufficiently slowly, and use the modified solution to show correct to $O(\epsilon^2)$ how the mean flow $\bar{u}(z, t)$ evolves in response to the waves. In the alternative reference frame that moves with the boundary, deduce the rate of decrease of mean-flow kinetic energy per unit horizontal area, correct to $O(\epsilon^2)$. Show that this is equal to W, and briefly explain why.

2 Consider 2-dimensional motion of a Boussinesq, continuously stratified, non-rotating, constant-N ideal fluid subject to an artificial force field $\mathbf{F}(x, z, t) = (F, 0, 0)$ that generates motion from an initial state of rest. Write down the linearized momentum equations that apply when $\mathbf{F}$ is weak enough. In terms of a streamfunction $\psi(x, z, t)$ such that $(u, w) = (\partial\psi/\partial z,\ -\partial\psi/\partial x)$, derive the corresponding equation for the y component of vorticity.

If the time-dependence of $\mathbf{F}$ contains only frequencies $\ll N$, how would you expect the horizontal scale of the response to compare with the vertical scale? What is the implication for the balance of terms (a) in the vertical momentum equation, and (b) in the vorticity equation?

Suppose that $F(x, z, t) = f(x, t)e^{imz}$ with $f(x, t)$ zero for $t < 0$, $\forall x$, and zero also for $|x| > a$, $\forall t$, where a is a constant. Suppose also that, within $|x| < a$ and for $t > 0$, f increases gradually toward a steady profile $f(x, \infty) \leqslant f_{\max}$, say, and that it does so slowly enough that $|\partial^2 f/\partial x\partial t| \ll |mNf_{\max}|$. Show that the vorticity equation, suitably approximated, then has the solution

$$\psi = \frac{-i\, e^{imz}}{2m} \int_{-\infty}^{t} \Big[f\big(x - c\,(t' - t), t'\big) + f\big(x + c\,(t' - t), t'\big) \Big]\, dt' \ ,$$

where $c = N/|m|$, and that the corresponding buoyancy-acceleration field is

$$\sigma = \frac{-icme^{imz}}{2} \int_{-\infty}^{t} \Big[f\big(x - c\,(t' - t), t'\big) - f\big(x + c\,(t' - t), t'\big) \Big]\, dt' \ .$$

Compute the vertical acceleration after making a change of variable that converts the range of integration to $(-\infty, 0)$, and deduce that the condition $|\partial^2 f/\partial x\partial t| \ll |mNf_{\max}|$ is enough to ensure the self-consistency of your approximation.

Show that the disturbance vanishes for $|x| > ct + a$, and draw graphs showing qualitatively the dependence of ψ and σ upon x.

3 Write down the shallow-water momentum and mass-conservation equations for a layer of homogeneous ideal fluid of depth $h(x, y, t)$ in a frame of reference rotating with constant angular velocity $(0, 0, \frac{1}{2}f)$. Allow for a sloping bottom boundary $z = b(x, y)$, taking care to distinguish between the layer depth h and the free-surface elevation $\zeta(x, y, t)$.

Derive the equation for the vertical component of absolute vorticity, $q_a(x, y, t)$. Deduce Rossby's exact potential-vorticity conservation theorem, explaining why it is h and not ζ that enters into the expression for the potential vorticity. Why is there no term in $w\partial/\partial z$?

Assume now that $q \ll f$ where q is the relative vorticity, $q = q_a - f$, and that $\zeta \ll h_{00}$ and $b \ll h_{00}$ where $h = h_{00} + \zeta - b$ with h_{00} constant. Derive an approximate expression for the potential vorticity, correct to the first order of small quantities, in which the leading term is f/h_{00}.

For small Rossby numbers show that the horizontal velocity field can be represented by a streamfunction $\psi(x, y, t)$, to be specified. Show that the approximate potential vorticity just derived is then proportional to

$$f + \frac{f}{h_{00}} b + \nabla_H^2 \psi - \frac{\psi}{L_R^2} \qquad (*)$$

where L_R is a constant to be specified. Specify also the meaning of the symbol ∇_H^2.

Consider a thought-experiment in which the initial state is one of relative rest, with $\zeta = 0$ everywhere, and with a lower boundary specified by

$$b = b(y) = \begin{cases} \epsilon a, & y > a \\ \epsilon y, & |y| < a \\ -\epsilon a, & y < -a \end{cases}$$

where a and ϵ are positive constants with $a\epsilon/h_{00} \ll 1$. Assume that a breaking Rossby wave perfectly mixes the potential vorticity in the central zone $|y| < a$, so that the x-averaged potential vorticity is unaffected outside $|y| < a$ but now has the constant value f/h_{00} within $|y| < a$. Sketch the *change* in $(*)$ as a function of y. Deduce that within $|y| < a$ the resulting velocity profile $u(y)$ is given by

$$u = \frac{\epsilon f L_R^2}{h_{00}} \left\{ -1 + \left(1 + \frac{a}{L_R}\right) \exp\left(\frac{-a}{L_R}\right) \cosh\left(\frac{y}{L_R}\right) \right\} .$$

In the case $L_R \gg a$, show further that

$$u = \frac{\epsilon f L_R^2}{h_{00}} \left\{ \frac{y^2 - a^2}{2L_R^2} + O\left(\frac{a}{L_R}\right)^3 \right\} \qquad (\, |y| < a\,) .$$

In the case $L_R \ll a$, sketch the profiles of $u(y)$ and $\zeta(y)$, for $|y| < 2a$.

4 Write an essay on the conservation and invertibility principles for potential vorticity in a stably-stratified ideal fluid, in a rotating reference frame having constant angular velocity $\mathbf{\Omega} = (0, 0, \frac{1}{2}f)$. The essay should cover the following:

(i) a statement and derivation of Ertel's theorem for the *Rossby–Ertel potential vorticity* $\rho^{-1}(2\mathbf{\Omega} + \nabla \times \mathbf{u}) \cdot \nabla\alpha$ (you should explain the meaning of the notation);

(ii) a statement and derivation of the conservation theorem for the *quasi-geostrophic potential vorticity*, Q say, for an unbounded Boussinesq, stably-stratified ideal fluid with constant buoyancy frequency N, and a demonstration that in appropriately stretched coordinates Q is the Laplacian of a certain streamfunction, up to an additive constant;

(iii) an explanation of what is meant by inversion in the unbounded quasi-geostrophic case, including a clear statement of the mathematical problem to be solved;

(iv) an illustrative example in which Q is piecewise constant, with the value f outside a spherical region in the stretched coordinates, and a slightly greater value $f + Q'$ inside that region;

(v) a brief indication of why, in that example, the relative vorticity in the equatorial plane of the spherical region has opposite signs inside and outside the region;

(vi) a brief discussion of why the conservation and invertibility principles help to simplify the task of understanding stratified, rotating flows, with comments on the implications of the single time derivative in the evolution equation.

END OF PAPER