UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 1 June, 2006 1.30 to 4.30

PAPER 76

NONLINEAR CONTINUUM MECHANICS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Prove Nanson's formula,

$$\mathbf{dS} = J\mathbf{F}^{-T}\mathbf{dS}_0,$$

relating an element of area $\mathbf{dS}_0$ to the element of area $\mathbf{dS}$ into which it is deformed under a deformation whose gradient is $\mathbf{F}$. Deduce the relation between components σ_{ji} of Cauchy stress and P_{Ii} of nominal stress. Define Kirchhoff stress (with components τ_{ji}) and relate this to nominal stress.

Given that the rate of working of the stress, per unit reference volume, is $P_{Ii}\dot{F}_{iI}$, show that it is also expressible as $\tau_{ji}D_{ij}$, where $\mathbf{D}$ denotes the Eulerian strain-rate.

State what is meant by a stress measure $\mathbf{T}$, conjugate to a strain measure $\mathbf{E}$. Show that the stress measure $\mathbf{T}^{(2)}$ conjugate to the strain measure $\mathbf{E}^{(2)} = (1/2)(\mathbf{F}^T\mathbf{F} - \mathbf{I})$ satisfies the relation

$$\mathbf{T}^{(2)} = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T}.$$

Find a corresponding expression for the stress measure $\mathbf{T}^{(-2)}$ conjugate to $\mathbf{E}^{(-2)} = (1/2)(\mathbf{I} - \mathbf{F}^{-1}\mathbf{F}^{-T})$. Interpret both stress measures in relation to a coordinate net which deforms with the body.

2 Express in integral form the first law of thermodynamics (the energy balance) relative to initial (Lagrangian) coordinates, for a body of initial density ρ_0 and internal energy per unit mass u, subjected to heat input r and body force $\mathbf{g}$ per unit mass, nominal surface tractions $n_I^0 P_{Ii}$ and (outward) heat flux $n_I^0 q_I^0$. Give the corresponding integral form of the entropy inequality. Deduce the local form of the energy balance,

$$\rho_0 \dot{u} = \rho_0 r - \frac{\partial q_I^0}{\partial X_I} + P_{Ii} \dot{F}_{iI},$$

where F_{iI} denote the components of the deformation gradient $\mathbf{F}$. Give also the local form of the entropy inequality.

Express these local relations in terms of the free energy density $\psi = u - \theta\eta$ in place of u, where θ denotes temperature and η is entropy per unit mass. Given that ψ is expressed as a function of $\mathbf{F}$, the temperature θ and a set of internal variables $\{\xi_r\}$, deduce the constitutive relations

$$P_{Ii} = \rho_0 \frac{\partial \psi}{\partial F_{iI}}, \quad \eta = -\frac{\partial \psi}{\partial \theta}$$

and the remaining inequality which involves the dissipative term $f_r \dot{\xi}_r$, where $f_r = -\partial\psi/\partial\xi_r$. Show also that

$$\rho_0 \theta \dot{\eta} = \rho_0 r - \frac{\partial q_I^0}{\partial X_I} + f_r \dot{\xi}_r.$$

Now assume that the material is constrained so that $\phi(\mathbf{F}) = h(\theta)$. Deduce that now

$$P_{Ii} = \rho_0 \frac{\partial \psi}{\partial F_{iI}} + q \frac{\partial \phi}{\partial F_{iI}},$$

where q is an undetermined scalar, and give the corresponding relation for the entropy.

Linearize about a stress-free state for which $\mathbf{F} = \mathbf{I}$, $\theta = \theta_0$ (so that $h(\theta_0) = 1$), $\eta = \eta_0$ and $q = 0$. Disregarding (as usual) the distinction between Lagrangian and Eulerian coordinates, deduce the equations of linear thermoelasticity, subject to the constraint of incompressibility (so that $\phi(\mathbf{F}) = \det(\mathbf{F})$)

$$\sigma_{ji} = C_{jilk} \frac{\partial u_k}{\partial x_l} + \beta_{ji}(\theta - \theta_0) + q\delta_{ji},$$

$$\rho_0(\eta - \eta_0) = C_e(\theta - \theta_0) - \beta_{ji} \frac{\partial u_i}{\partial x_j} + q h'(\theta_0),$$

expressing the constants C_{jilk}, β_{ji} and C_e in terms of ψ.

3 A circular cylinder with generators aligned with the 3-axis has, before deformation, radius A and height H. It is composed of incompressible, isotropic hyperelastic material and has energy function $W(\lambda_1, \lambda_2, \lambda_3)$ in terms of the principal stretches λ_r, $r = 1, 2, 3$ $(\lambda_1\lambda_2\lambda_3 = 1)$. It is subjected to axial stretch and torsion, so that

$$\begin{aligned} x_1 &= \lambda^{-1/2}[X_1 \cos(\alpha X_3) - X_2 \sin(\alpha X_3)], \\ x_2 &= \lambda^{-1/2}[X_1 \sin(\alpha X_3) + X_2 \cos(\alpha X_3)], \\ x_3 &= \lambda X_3. \end{aligned}$$

Show that the principal stretches are given by

$$\begin{aligned} \lambda_1^2 &= \lambda^{-1}, \\ \lambda_{2,3}^2 &= \frac{1}{2}\left\{\lambda^{-1}(1+\alpha^2R^2) + \lambda^2 \pm \left[(\lambda^{-1}(1+\alpha^2R^2)+\lambda^2)^2 - 4\lambda\right]^{1/2}\right\}, \end{aligned}$$

where $R^2 = X_1^2 + X_2^2$.

By equating the rate of working of the applied loads, per unit initial height, to $M\dot{\alpha} + N\dot{\lambda}$, show that

$$M = 2\pi \frac{\partial}{\partial \alpha} \int_0^A R\, dR\, W(\lambda_1, \lambda_2, \lambda_3)$$

and give the corresponding expression for N. Evaluate M and N, when the cylinder is composed of Mooney material with energy function

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{2}\mu_1\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) - \frac{1}{2}\mu_2\left(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3\right),$$

where μ_1 and μ_2 are constants.

4 A hyperelastic material occupying a domain Ω prior to deformation has energy function $W(\mathbf{F})$ and mass density ρ_0 per unit undeformed volume and is subject to a constraint $\phi(\mathbf{F}) = 0$. It is maintained in equilibrium under dead-load body force $\mathbf{g}$ and nominal surface tractions $T_i = N_I P_{Ii}$. Show that the equilibrium is stable against a small time-dependent perturbation $\delta\mathbf{x}$ if

$$\int_\Omega \left(\frac{\partial^2 W}{\partial F_{iI} \partial F_{jJ}} + q \frac{\partial^2 \phi}{\partial F_{iI} \partial F_{jJ}} \right) \delta F_{iI} \delta F_{jJ} \, d\mathbf{X} > 0$$

for all admissible $\delta\mathbf{F}$ not identically zero, where q is the multiplier in the definition of the stress, in the equilibrium configuration.

Consider now a unit cube (when undeformed), composed of neo-Hookean material with energy function $W(\mathbf{F}) = (\mu/2)(F_{iI}F_{iI} - 3)$, subjected to dead-load all-round tensile loading T (so that $P_{Ii} = T$ if $i = I$ and $P_{Ii} = 0$ otherwise). By considering possible equilibrium configurations

$$\mathbf{F} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{pmatrix},$$

show that

$$\text{either } \lambda = 1 \text{ or } T/\mu = \lambda + \lambda^{-1/2}.$$

Show also that

$$q = T\lambda - \mu\lambda^2.$$

Deduce that there are two solutions of the given form, with $\lambda \neq 1$, so long as $T/\mu > 3(2^{-2/3})$.

Investigate the stability of each of these equilibria, against uniform perturbations of the form $\delta\mathbf{F} = \mathrm{diag}(\delta\lambda, -(1/2)\lambda^{-3/2}\delta\lambda, -(1/2)\lambda^{-3/2}\delta\lambda)$. Show that the solution $\lambda = 1$ is stable (against such a perturbation) if $T/\mu < 2$. Show that, when there are three solutions, two are stable and one is unstable. Identify the value of T/μ corresponding to the point of bifurcation.

5 Differentiate the stress measure $\mathbf{T} = \mathbf{F}^T \boldsymbol{\tau} \mathbf{F}$ to derive the "lower convected" time-derivative of the Kirchhoff stress $\boldsymbol{\tau}$,

$$\frac{\delta_l \boldsymbol{\tau}}{\delta_l t} = \dot{\boldsymbol{\tau}} + \mathbf{L}^T \boldsymbol{\tau} + \boldsymbol{\tau} \mathbf{L},$$

where $\mathbf{L}$ denotes the Eulerian rate of deformation. Check explicitly that this stress-rate is objective.

The "lower convected Oldroyd" fluid is incompressible and has constitutive relation $\boldsymbol{\sigma} = \boldsymbol{\sigma}^d - p\mathbf{I}$, where

$$\frac{\delta_l \boldsymbol{\sigma}^d}{\delta_l t} + \frac{\boldsymbol{\sigma}^d}{\tau} = \frac{2\mu}{\tau}\mathbf{D} + 2\mu_r \frac{\delta_l \mathbf{D}}{\delta_l t},$$

the Eulerian strain-rate being $\mathbf{D}$. Show that, equivalently,

$$\boldsymbol{\sigma}^d(t) = 2\mu_r \mathbf{D}(t) + \frac{2(\mu - \mu_r)}{\tau} \int_{-\infty}^{t} e^{-(t-t')/\tau} \mathbf{F}^{-T}(t) \mathbf{F}^T(t') \mathbf{D}(t') \mathbf{F}(t') \mathbf{F}^{-1}(t)\, dt'.$$

Give this relation explicitly, for the time-dependent simple shear deformation

$$\mathbf{F}(t) = \begin{pmatrix} 1 & \gamma(t) \\ 0 & 1 \end{pmatrix},$$

disregarding the trivial 3-components. Evaluate the integrals for the steady-state case $\gamma(t) = \dot{\gamma} t$, where $\dot{\gamma}$ is constant, and hence deduce the normal stress difference $\sigma_{11} - \sigma_{22}$.

6 Consider the infinitesimal deformation of incompressible, isotropic, elasto-plastic material, obeying the non-hardening von Mises yield criterion

$$\sigma'_{ij}\sigma'_{ij} = 2k^2$$

(where k is a constant) and the associated flow rule. Show that it is consistent, under the plane strain condition that displacement $\mathbf{u}$ has the form $(u_1(x_1, x_2), u_2(x_1, x_2), 0)$, to take both elastic and plastic parts of the strain component e_{33} equal to zero.

Show that, for such deformation, the yield condition is satisfied identically by taking

$$\sigma_{11} = -p + k\sin(2\phi), \quad \sigma_{22} = -p - k\sin(2\phi), \quad \sigma_{12} = -k\cos(2\phi).$$

[*The signs are chosen to facilitate the solution of the problem to follow.*] Define "α-lines" and "β-lines" and show that

$$\begin{aligned} p - 2k\phi &= \text{ constant on an } \alpha\text{-line}, \\ p + 2k\phi &= \text{ constant on a } \beta\text{-line}. \end{aligned}$$

A plane strain specimen of material of the type described contains a V-notch of semi-angle γ, whose line of symmetry is the x_1-axis. It is subjected to tensile loading which is symmetric about this axis. Calculate the traction components on one surface of the notch, in the plastic region, and deduce the values of p and ϕ there. Show that, on the x_1-axis just ahead of the notch, $\sigma_{22} = k(2 + \pi - 2\gamma)$.

END OF PAPER