UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Wednesday 7 June, 2006   9 to 12

PAPER 74

STELLAR AND PLANETARY MAGNETIC FIELDS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a magnetic field $\mathbf{B}(\mathbf{x}, t)$ acted on by a solenoidal velocity field $\mathbf{u}(\mathbf{x}, t)$ confined to a sphere $\mathcal{V}$ of radius a surrounded by insulator. The diffusivity in $r < a$ takes the constant value η.

(i) Prove that $P \equiv \mathbf{B} \cdot \mathbf{x}$ satisfies the equation

$$\frac{\partial P}{\partial t} + \mathbf{u} \cdot \nabla P = \mathbf{B} \cdot \nabla Q + \eta \nabla^2 P,$$

in $r < a$, where $Q = \mathbf{u} \cdot \mathbf{x}$. What is the equation satisfied by P in $r > a$, and what jump conditions should be applied at $r = a$?

(ii) Derive the result

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{V}} P^2 dV \leqslant \max_{\mathcal{V}}|Q| \int_{\mathcal{R}^3} |\mathbf{B} \cdot \nabla P| dV - \eta \int_{\mathcal{R}^3} |\nabla P|^2 dV$$

and use the Schwartz inequality to show that a working dynamo (with $\int_{\mathcal{V}} P^2 dV$ not decaying to zero) can exist only if

$$(\max_{\mathcal{V}}|Q|)^2 \geqslant \eta^2 \frac{\int_{\mathcal{R}^3} |\nabla P|^2 dV}{\int_{\mathcal{R}^3} |\mathbf{B}|^2 dV}$$

(iii) Given that the poloidal field $\mathbf{B}_P$ satisfies the relation

$$\int_{\mathcal{R}^3} |\mathbf{B}_P|^2 dV = \int_{\mathcal{R}^3} \nabla(\mathcal{L}^{-2} P) \cdot \nabla P \, dV,$$

where $\mathcal{L}^2 = -\mathbf{x}^2\nabla^2 + (\mathbf{x} \cdot \nabla)^2$ is the angular momentum operator with eigenvalues $l(l+1)$, $l = 0, 1, \ldots$, show that $\int_{\mathcal{R}^3} |\mathbf{B}_P|^2 dV \leqslant \frac{1}{2} \int_{\mathcal{R}^3} |\nabla P|^2 dV$, and hence derive an inequality for a working dynamo involving Q and the ratios of poloidal and total magnetic energy.

2 Consider the mean emf due to a steady solenoidal velocity field $\mathbf{u}(\mathbf{x})$ acting on a uniform magnetic field at small magnetic Reynolds number R_m. The velocity field is monochromatic, so that $\nabla^2\mathbf{u} = -\mathbf{u}$. The induction equation takes the form

$$0 = \mathbf{B} \cdot \nabla\mathbf{u} + \nabla \times (\mathbf{u} \times \mathbf{b}) + \frac{1}{R_m}\nabla^2\mathbf{b},$$

where $\mathbf{B}$ is a constant vector, and $\mathbf{b}$ is the induced field. Show that the mean emf $\mathcal{E} = \overline{\mathbf{u} \times \mathbf{b}}$, where the overbar denotes an average over all space, can be written in the form

$$\mathcal{E} = R_m\mathcal{E}^{(1)} + R_m^2\mathcal{E}^{(2)} + \ldots = R_m\overline{\mathbf{u} \times \mathbf{B} \cdot \nabla\mathbf{u}} + R_m^2\overline{\mathbf{u} \times \nabla \times (\mathbf{u} \times \mathbf{B} \cdot \nabla\mathbf{u})} + \ldots \ .$$

Show, without using Fourier decomposition, that if $\mathcal{E}_i^{(p)}$ can be written $\alpha_{ij}^{(p)}B_j$, then $\alpha^{(1)}$ is symmetric.

Derive the result

$$\mathcal{E}_i^{(2)} = -\overline{\frac{\partial u_j}{\partial x_i}(\mathbf{u} \times \mathbf{B} \cdot \nabla\mathbf{u})_j} \ ,$$

and hence , or otherwise, show that $\alpha^{(2)}$ is antisymmetric.

3 A layer of Boussinesq fluid (constant density and material properties) fills the region $0 < z < d$. A uniform constant horizontal field $(0, B_0, 0)$ is imposed. The boundary conditions imposed at $z = 0, d$ are: constant temperature ($T(0) = T_0 + \Delta T$, $T(d) = T_0$ (so that the temperature perturbation θ vanishes at $z = 0, d$); stress-free ($u_z = 0 = (\partial/\partial z)(u_x, u_y)$); and the field perturbation $\mathbf{b}$ is horizontal, with $(\partial/\partial z)(b_x, b_y) = 0$.

(i) Write down the equations describing linearised disturbances, show that they may be cast in the dimensionless form

$$\frac{1}{\sigma}\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \zeta Q \frac{\partial \mathbf{b}}{\partial y} + R\theta\hat{\mathbf{z}} + \nabla^2 \mathbf{u}\,,$$

$$\frac{\partial \theta}{\partial t} = u_z + \nabla^2 \theta\,,$$

$$\frac{\partial \mathbf{b}}{\partial t} = \frac{\partial \mathbf{u}}{\partial y} + \zeta \nabla^2 \mathbf{b}\,,$$

$$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0\,,$$

and define the parameters R, Q, σ, ζ in terms of physical quantities.

(ii) Seek solutions in the form $\theta, u_z, b_z \propto e^{st+ikx+imy}\sin\pi z$. Assuming that $\mathbf{u}, \mathbf{b}$ may be expressed as the poloidal vectors $\nabla \times \nabla \times \phi\hat{\mathbf{z}}$, $\nabla \times \nabla \times \chi\hat{\mathbf{z}}$, find an equation for the growth rate s as a function of R, Q, σ, ζ, k and m.

(iii) Now concentrate on the onset of steady disturbances, so that $s = 0$. Show that the smallest value of R for such disturbances has $m = 0$ and is independent of Q. Give a physical explanation for this result.

(iv) Notwithstanding the result of (iii), look for two-dimensional disturbances varying along the field, with $k = 0$, $m \neq 0$. Write down an equation connecting R, Q and m for steady solutions in this case, and show that for fixed large values of Q, the smallest value of R is achieved when $m^4 \approx \pi^6/Q$. Give an expression for this smallest value correct up to and including terms of order unity.

4 Uniform magnetic field $(0, B_1, 0)$ fills the region $z < 0$, while for $z > 0$, the field is also uniform, but is $(B_2, 0, 0)$, where $|B_2| < |B_1|$. Gravity $\mathbf{g} = -g\hat{\mathbf{z}}$. The temperature T_0 is uniform, and the fluid may be taken as an isothermal perfect gas with gas constant $\mathcal{R}$. The density at $z = 0^-$ is ρ.

Find the hydrostatic pressures and densities $p_i(z), \rho_i(z)$, $i = 1, 2$, in $z < 0$, $z > 0$ respectively.

On the assumption of irrotational motion, find an equation for the growth rate s of isothermal disturbances to the basic equilibrium which are proportional to $\exp(st + ikx + imy)$, and determine the maximum growth rate, and the values of k, m for which it occurs. It may be assumed for this calculation that the vertical extent of any instability away from the interface is much less than the scale height $\mathcal{R}T_0/g$, so that pressures and densities may be assumed to be constant and equal to their values on either side of the interface.

Give conditions on the physical quantities which ensure that the vertical extent of the disturbance from the interface for the maximum growth rate mode is much smaller than any hydrostatic scale height.

END OF PAPER