UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 9 June, 2006 1.30 to 4.30

PAPER 73

GALAXIES AND DARK MATTER

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The observed phase space density of binary stars depends on orbital eccentricity as $F(e) \propto e^2$. Jeans considered the Boltzmann distribution $dN \propto \exp(-E/\sigma)$, N particle number, E energy, to show this predicts $F(e) \propto e^2$, and concluded binary stars are a relaxed system, with age $\sim 10^{13}$ years, which he deduced to be the age of the Universe.

(i) Show the relaxation time for a stellar system like the solar neighbourhood is

$$T_R = \frac{3}{16\pi\sqrt{2}} \quad \frac{\sigma^3}{\mathrm{NG}^2\mathrm{m}^2 ln\Lambda} \ ,$$

where σ is a representative 1-D velocity, m a star mass, N the number of stars, and Λ is to be explained. Estimate its value for the solar neighbourhood.

(ii) Now consider a phase space distribution of binaries

$$dN = F(E)dx\ dy\ dz\ dp_x\ dp_y\ dp_z \ ,$$

where $x,\ y,\ z$ are spatial coordinates, and p_i the momenta.

Perform a canonical transformation to Delaunay elements, $L,\ G,\ H$, and corresponding angular variables $l,\ g,\ h$ where

$$\begin{aligned} L^2 &= \gamma m a \\ G^2 &= \gamma m a(1-e^2) \\ H^2 &= \gamma m a(1-e^2)(\cos^2 i) \end{aligned}$$

where a and e are the semi-major axis of and eccentricity of the orbit, i its inclination and m the system total mass, γ is the Newtonian gravitational constant. Deduce the ranges of the independent orthogonal variables $L,\ G,\ H,\ l,\ g,\ h$.

Show in this case the number of stars with eccentricity less than some value e is $N \propto e^2$, for any finite $F(E)$. Conclude that one cannot deduce equilibrium from an observed $F(e) \propto e^2$.

2 Show that the first velocity moment of the collisionless Boltzmann equation, for a static spherical system, can be written

$$\frac{\mathrm{d}}{\mathrm{d}r}(\lambda\sigma_r^2) + \frac{2\beta}{r}\lambda\sigma_r^2 = -\gamma\frac{\lambda \mathrm{L}(r)}{r^2} \ ,$$

where σ_r is the radial component of the velocity dispersion, $\beta = 1 - (\sigma_t^2/\sigma_r^2)$, with σ_t the tangential component, λ is the luminosity density, $\gamma =$ G times the mass to light ratio, and $\mathrm{L}(r)$ is the enclosed luminosity.

The surface brightness μ may be related to λ by

$$\lambda(r) = -\frac{1}{\pi}\int_r^\infty \frac{\mathrm{d}x}{(x^2-r^2)^{1/2}}\mu'(x) \ .$$

By considering the equivalent relation for the projected velocity dispersion, $\sigma_\mathrm{p}^2\mu$, show that the observables can be related to a function of coordinates and $\lambda\sigma_r^2$ by

$$\sigma_\mathrm{p}^2\mu - \gamma\int_r^\infty \frac{\mathrm{d}x}{(x^2-r^2)^{1/2}}\ \frac{r^2}{x^2}\gamma \mathrm{L} \equiv \mathrm{p}(r) \ .$$

From term by term inspection, show $\int_0^\infty\ 6\pi r\mathrm{p}(r)\mathrm{d}r = 0$ is equivalent to the virial theorem, so that

$$\gamma = \frac{3\int_0^\infty r\sigma_\mathrm{p}^2\mu\mathrm{d}r}{2\int_0^\infty r\lambda \mathrm{L}\mathrm{d}r} \ .$$

3 Consider a triaxial galaxy whose density distribution is stationary in a coordinate frame that rotates with angular frequency Ω about the x_3 axis, which coincides with a principal axis of the moment of inertia tensor I. Show that, at an instant when $I_{12} = 0$,

$$\frac{1}{2}\,\frac{d^2 I}{dt^2}$$

equals Ω^2 times the diagonal tensor which has components $(I_{22} - I_{11}),\ (I_{11} - I_{22}),\ (0)$ on the diagonal. From this show

$$\Omega^2 = -\frac{(W_{11} - W_{22}) + 2(T_{11} - T_{22}) + (\Pi_{11} - \Pi_{22})}{2(I_{11} - I_{22})}$$

and, for $T_{33} = 0$,

$$\frac{\nu_0^2}{\sigma_0^2} = (1 - \delta)\frac{W_{11} + W_{22}}{W_{33}} - 2$$

where $W_{ij},\ T_{ij}$, and Π_{ij} have their normal tensor-virial defintions, the mass weighted rotation velocity

$$V_0^2 \equiv \frac{2}{M}(T_{11} + T_{22})\ ,$$

the mass-weighted velocity dispersion

$$\sigma_0^2 = \frac{1}{2M}(\Pi_{11} + \Pi_{22})$$

and the velocity anisotropy parameter

$$\delta = 1 - \frac{2\Pi_3}{\Pi_1 + \Pi_2}\ .$$

For the case of an axi-symmetric oblate spheroidal galaxy, with true axis ratio $(1 - \epsilon_t)$ observed at inclination i, show that the observed rotation velocity V_r, velocity dispersion σ^2, and apparent axial ratio $(1 - \epsilon_a)$ are related to the true values by

$$V_r = V_0 \sin i$$
$$\sigma^2 = \sigma_0^2(1 - \delta\cos^2 i) \text{ and}$$
$$\epsilon_a(2 - \epsilon_a) = \epsilon_t(2 - \epsilon_t)\sin^2 i$$

4 (a) Consider an axisymmetric disk galaxy, described in an (R, ϕ, z) cylindrical coordinate system, with density profile $\nu(R)$ and gravitational potential Φ. Show the radial velocity moment (Jeans' equation) of the collisionless Boltzmann equation at $z = 0$ is

$$\frac{R}{\nu}\frac{\partial}{\partial R}(\nu \overline{v_R^2}) + R\frac{\partial(\overline{v_R v_z})}{\partial z} + \overline{v_R^2} - \overline{v_\phi^2} + R\frac{\partial \Phi}{\partial R} = 0 \ ,$$

where $v_R, \ v_\phi$, and v_z are velocities.

Discuss the application of this equation to the Milky Way Galaxy, in particular to explain the asymmetric drift of stellar motions.

(b) Explain how one may use Jeans' theorem to go from the density of a 1-D system

$$\nu(z) = \int_{-\infty}^{\infty} f_z(z, v_z) dv_z \ ,$$

to the distribution function

$$F_z(E_z) = \frac{1}{\pi}\int_{E_z}^{\infty} \frac{-d\nu/d\psi}{\sqrt{2(\psi - E_z)}} d\psi \ ,$$

where ψ is the potential. Identify all assumptions and requirements, including those which allow application of a 1-D analysis to a 3-D galaxy. Discuss the application of this equation to determination of the local disk mass density, and discuss its implications.

END OF PAPER